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Abstract: This paper addresses the stability analysis of a system of ordinary differential equations coupled with a classic heat equation using a Lyapunov approach. Relying on recent developments in the area of time-delay systems, a new method to study the stability of such a class of coupled finite/infinite dimensional systems is presented here. It consists in a Lyapunov analysis of the infinite dimensional state of the system using an energy functional enriched by the mean value of the heat variable. The main technical step relies on the use an efficient Bessel-like integral inequality on Hilbert space leading to tractable conditions expressed in terms of linear matrix inequalities. The results are then illustrated on academic examples and demonstrate the potential of this new approach.

Keywords: Heat equation, Lyapunov functionals, integral inequalities, LMI.

1. INTRODUCTION

The last decade has seen the emergence of number of papers concerning the stability or control of finite dimensional systems connected with infinite-dimensional systems (see Susto and Krstic (2010), Krstic (2009) and references therein). The applications are various: the finite dimensional systems can for instance represent a dynamic controller for a system modeled by a partial differential equation (PDE), see Krstic (2009) and references therein. One can also imagine PDEs modeling actuators or sensors' behavior and the goal is then to study the stabilization of a finite system of ordinary differential equations (ODEs) despite the infinite nature of actuators/sensors dynamics (as e.g. in time delay systems). Conversely, an ODE can model a component coupled to a phenomenon described by PDEs as in Daafouz et al. (2014).
The stability notion employed in the literature depends not only on the types of interconnections between the ODE and the PDE, the boundary conditions of the PDE but also on the norms used (see Tang and Xie (2011)). Classical approaches for studying the stability of such an interconnection rely firstly on discretization techniques leading to some finite dimensional systems to be studied. Another possibility is to model the overall system using the semi-group approach. It may lead, as in Fridman (2014), to Linear Operator Inequality to be solved. Nevertheless, this appealing approach remains limited since there are no available tools to solve these Linear Operator Inequalities and they should be solved by hand, see Fridman and Orlov (2009). Complicated to handle, these techniques are then confined to small dimensional ODE systems. Anyway, the choice of an efficient Lyapunov functional can be very complicated, except in a few specific cases, see Curtain and Zwart (1995) or Bastin and Coron (2015).

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In general, the chosen Lyapunov functional appears to be the sum of a classical Lyapunov function, for the finite dimensional part of the system, with the energy of the PDE part (see Prieur and Mazenc (2012), Papachristodoulou and Peet (2006)). This is the case for instance in Krstic (2009), where the control of a finite dimensional system connected to an actuator/sensor modeled by a heat equation with Neuman and Dirichlet boundary conditions is considered. The authors adopt the backstepping method employed originally in the case of the transport (or delay) equation. The resulting feedback system is equivalent to a finite dimensional exponentially stable system cascaded with a heat condition and the choice of an appropriate Lyapunov function as a sum of the energy of the heat equation and a classical quadratic Lyapunov function for the finite dimensional system allows to prove the exponential stability of the overall system. In Papachristodoulou and Peet (2006), a very general structure for the Lyapunov function is given for a parabolic type of PDE. An optimization relying on Sum of Squares techniques are then proposed to compute the parameters of the Lyapunov function structure.
Here, we propose to analyze the stability of a heat equation coupled with a system of ODEs by constructing an adequate Lyapunov functional, reducing the conservatism of the more classical approaches. Unlike some papers of the literature, we aim at proving the exponential stability using a Lyapunov function which is enriched with an extra term depending of the mean value of the heat state. The main idea relies in the introduction of some cross terms in the Lyapunov functional, idea that have been fruitful in a non linear finite dimensional context as in Sepulchre et al. (1997). While the introduction of a cross term does not introduce difficulties related to the definition of a candidate Lyapunov functional, this term still implies that some inequalities must be evaluated when deriving this functional along the trajectories of the system. The use of

Bessel inequalities allows then to estimate these integral terms and obtain some numerical tractable inequalities. Notice that this approach is similar in its spirit to Ahmadi et al. (2016) where the stability of an interconnected system is tackled via an input output approach and relies heavily on some inequalities which are evaluated thanks to Sum Of Square techniques but at a price of an increasing numerical complexity. Another similar approach has been also studied in the case of the interconnection between an ODE and a transport equation, see Baudouin et al. (2016), that mimics actually a time delay system as in Seuret and Gouaisbaut (2015).

Notation. $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote the sets of positive integers, positive reals, n-dimensional vectors and $n \times m$ matrices ; the Euclidean norm writes $|\cdot|$. For any matrix A in $\mathbb{R}^{n\times n}$, we denote $\mathrm{He}(A) = A + A^\top$ where $A^\top$ is a transpose matrix. For a partitioned matrix, the symbol $*$ stands for symmetric blocks, I is the identity and 0 the zero matrix. For any $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ means that P is symmetric positive definite. The partial derivative with respect to x is denoted $\partial_x u = \frac{\partial u}{\partial x}$ (while the time derivative is $\dot{X} = \frac{dX}{dt}$). Finally, $L^2(0,1)$ is the Hilbert space of square integrable functions over $]0,1[$ with values in $\mathbb{R}$ and one writes $\|z\|^2 = \int_0^1 |z(x)|^2\,dx = \langle z, z\rangle$, and we also define the Sobolev space $H^1(0,1) = \{z \in L^2(0,1), \partial_x z \in L^2(0,1)\}$ and its norm by $\|z\|^2_{H^1(0,1)} = \|z\|^2 + \|\partial_x z\|^2$.

2. FORMULATION OF THE PROBLEM

Let us detail our coupling of a heat equation with a system of ODEs and how we can then analyze its stability.

2.1 A coupled system

We consider the following coupling of a finite dimensional system in the vectorial variable X with a heat partial differential equation in the scalar variable u:

$$\begin{cases} \dot{X}(t) &= AX(t) + Bu(1,t) & t > 0, \\ \partial_t u(x,t) &= \gamma \partial_{xx} u(x,t) & x \in (0,1), t > 0, \\ u(0,t) &= CX(t), & t > 0 \\ \partial_x u(1,t) &= 0, & t > 0. \end{cases} \quad (1)$$

The state of the system is the pair $(X(t), u(t))$ that belongs to $\mathbb{R}^n \times H^1(0,1)$ and satisfies the compatible initial datum $(X(0), u(\cdot,0)) = (X^0, u^0) \in \mathbb{R}^n \times H^1(0,1)$. The thermal diffusivity is denoted $\gamma \in \mathbb{R}_+$, and the matrix $A \in \mathbb{R}^{n\times n}$, the vectors $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ are constant.

The goal of this article is the study of stability of the complete infinite dimensional system (1). It will be performed thanks to the construction of a Lyapunov functional based on the weighed classical energy of the full system in (X,u) enriched by a quadratic term built on the mean value $\bar{u}$ of the distributed state u. The main contribution of this paper is to present a method inspired from the recent developments on the stability analysis of time-delay systems, which is a particular class of infinite dimensional systems, in order to provide efficient tools for such an analysis.

2.2 Existence and regularity of the solutions

For the record, the equation $\partial_t v - \gamma\partial_{xx} v = 0$ set in $(0,1) \times \mathbb{R}^+$ of unknown $v = v(x,t)$ is a classic heat PDE and if the boundary data are of Dirichlet homogeneous type (*i.e.* $v(0,t) = v(1,t) = 0$) and the initial datum $v(\cdot,0) = v^0$ belongs to $H^1_0(0,1) = \{z \in H^1(0,1), z(0) = z(1) = 0\}$, then it has a unique solution u satisfying

$$v \in \mathcal{C}(\mathbb{R}_+; H^1_0(0,1)) \cap L^2(0,+\infty; H^2(0,1))$$
$$\partial_t v \in L^2(0,+\infty; L^2(0,1)),$$

see e.g. Brezis (1983).

In this paper, we deal with a coupled system for which the existence and regularity of the solution (X,u) stems from a Galerkin method (see e.g. Evans (2010)). System (1) admits indeed a unique solution (X,u) in the space

$$C^1(\mathbb{R}_+;\mathbb{R}^n) \times \left(\mathcal{C}(\mathbb{R}_+; H^1(0,1)) \cap L^2(0,+\infty; H^2(0,1))\right).$$

Let us define here the total energy of system (1) by:

$$E(X(t),u(t)) = |X(t)|^2_n + \|u(t)\|^2_{L^2(0,1)} + \|\partial_x u(t)\|^2_{L^2(0,1)}.$$

The Galerkin method gives after some classic computations that there exists a constant $K > 0$ depending only on A, B and C such that $\dot{E}(X(t),u(t)) \leq KE(X(t),u(t))$.

2.3 Existence and unicity of equilibrium

Before discussing about stability, a first natural step is to define the equilibrium of system (1).

Proposition 1. An equilibrium $(X_e, u_e) \in \mathbb{R}^n \times H^1(0,1)$ of system (1) verifies

$$(A + BC)X_e = 0, \text{ and } u_e \equiv CX_e. \quad (2)$$

Moreover if the matrix $A + BC$ is non singular, then the equilibrium of system (1) is unique and is

$$X_e = 0, \text{ and } u_e \equiv 0. \quad (3)$$

Proof : An equilibrium $(X_e, u_e) \in \mathbb{R}^n \times H^1(0,1)$ of (1) verifies the following set of equations:

$$\begin{cases} AX_e + Bu_e(1) = 0, & \\ \partial_{xx} u_e(x) = 0, & \forall x \in [0,1], \\ u_e(0) = CX_e, & \\ \partial_x u_e(1) = 0. & \end{cases} \quad (4)$$

From the second equation, the function u_e is a polynomial function in the space variable x of degree less than 1. Then, the boundary conditions in the third and fourth equation of (4) requires that u_e have the following form:

$$u_e(x) = CX_e, \qquad \forall x \in [0,1].$$

Re-injecting $u_e(1)$ in the first equation of (4) leads to the equation $(A + BC)X_e = 0$. Therefore, an equilibrium is any point in the kernel of the matrix $A + BC$. Moreover, the unicity of the equilibrium $(0,0) \in \mathbb{R}^n \times H^1(0,1;\mathbb{R})$ is guaranteed if the matrix $A + BC$ is non singular. $\square$

2.4 A Lyapunov functional for the coupled system

Our objective is to construct a Lyapunov functional in order to narrow the proof of the stability of the complete infinite dimensional system (1) to the resolution of Linear Matrix Inequalities (LMI). First of all, let us recall the notion of exponential stability which will be used in the sequel.

Definition 1. System (1) is said to be exponentially stable if for all initial conditions $(X^0, u^0) \in \mathbb{R}^n \times H^1(0,1)$, there exist $K > 0$ and $\delta > 0$ such that

$$E(X,u) \leq Ke^{-\delta t}\left(|X^0|^2_n + \|u^0\|^2_{H^1(0,1)}\right), \forall t > 0. \quad (5)$$

In general, the Lyapunov function employed to determine the exponential stability is the sum of a quadratic function of X and the energy of the heat equation. Here, hoping to reduce the conservatism of this decoupled approach, we would like to track the stability of the system through LMIs and our idea is to enrich slightly the classical total energy E of the system by the use of a first order approximation of the state in order to build a new candidate Lyapunov functional V.

We introduce a finite dimensional vector of size $n+1$, composed by the state of the ODE system X and the mean value $\bar{u}(t) = \int_0^1 u(x,t)\, dx$ of the distributed state u.

Inspired by the Lyapunov(-Krasovskii) functional, provided in Seuret and Gouaisbaut (2015), we construct a candidate Lyapunov functional for system (1) of the form:

$$V(X(t),u(t)) = \begin{bmatrix} X(t) \\ \bar{u}(t) \end{bmatrix}^\top \begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \begin{bmatrix} X(t) \\ \bar{u}(t) \end{bmatrix} + \alpha \int_0^1 |u(x,t)|^2\, dx + \beta \int_0^1 |\partial_x u(x,t)|^2\, dx, \quad (6)$$

where $P \in \mathbb{S}^n_+$, $Q \in \mathbb{R}^n$ and $T \in \mathbb{R}$ have to be determined.

Remark 1. As a shorthand notation, we will omit, from now on, the time argument 't'. Therefore, in the sequel, functions $X, u, \bar{u}$ or $\partial_x u$ denote implicit functions of time.

The objective of this paper is to provide simple and tractable conditions for assessing exponential stability of system (1) based on the Lyapunov functional (6) provided above. More particularly, we aim at ensuring that there exist some positive reals $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that

$$\frac{d}{dt}V(X,u) \leq -\varepsilon_1 E(X,u) \quad (7)$$

$$\varepsilon_2 E(X,u) \leq V(X,u) \leq \varepsilon_3 E(X,u), \quad (8)$$

where the notation $\frac{d}{dt}V(X,u)$ denotes the time derivative of the functional V along the trajectories of system (1).

Indeed, on the one hand, its suffices to notice that we obtain directly from (8) and (7), $\forall t \geq 0$,

$$\frac{d}{dt}V(X,u) + \frac{\varepsilon_1}{\varepsilon_3}V(X,u) \leq 0$$

so that $\frac{\mathrm{d}}{\mathrm{d}t}\left(V(X,u)e^{\varepsilon_1 t/\varepsilon_3}\right) \leq 0$ and integrating the previous expression in time, we get $V(X,u) \leq V(X^0,u^0)e^{-\varepsilon_1 t/\varepsilon_3}$. On the other hand, with the help of (8), we obtain

$$\begin{aligned} \varepsilon_2 E(X,u) &\leq V(X,u) \leq V(X^0,u^0)e^{-\varepsilon_1 t/\varepsilon_3} \\ &\leq \varepsilon_3 E(X^0,u^0)e^{-\varepsilon_1 t/\varepsilon_3}, \end{aligned}$$

allowing to conclude (5) with $\delta = \varepsilon_1/\varepsilon_3$.

2.5 First insights on the derivative of V

In the following section, conditions for exponential stability of the origin of system (1) will be obtained using the LMI framework. As an introductory computation, let us give here the time derivative of V along the trajectories to derive inequality (7). On the one hand, using the equations in system (1), we have :

$$\frac{d}{dt}\begin{bmatrix} X \\ \bar{u} \end{bmatrix} = \begin{bmatrix} AX + Bu(1) \\ -\gamma\partial_x u(0) \end{bmatrix}$$

so that we can calculate, still omitting the variable t,

$$\frac{d}{dt}\left(\begin{bmatrix} X \\ \bar{u} \end{bmatrix}^\top \begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \begin{bmatrix} X \\ \bar{u} \end{bmatrix}\right) = \begin{bmatrix} X \\ u(1) \\ \partial_x u(0) \\ \bar{u} \end{bmatrix}^\top \begin{bmatrix} \mathrm{He}(PA) & PB & -\gamma Q & A^\top Q \\ * & 0 & 0 & B^\top Q \\ * & * & 0 & -\gamma T \\ * & * & * & 0 \end{bmatrix} \begin{bmatrix} X \\ u(1) \\ \partial_x u(0) \\ \bar{u} \end{bmatrix}.$$

This first calculation suggests the definition of an augmented vector of size $n+3$ given by

$$\xi = \begin{bmatrix} X^\top u(1) \ \partial_x u(0) \ \bar{u} \end{bmatrix}^\top. \quad (9)$$

On the other hand, using the heat equation in (1), and an integration by parts, we get both

$$\begin{aligned} \alpha\frac{d}{dt}\int_0^1 |u(x)|^2\, dx &= 2\alpha\int_0^1 u(x)\partial_t u(x)\, dx \\ &= -2\alpha\gamma\int_0^1 |\partial_x u(x)|^2\, dx + 2\alpha\gamma\left[u\partial_x u\right]_0^1 \\ &= -2\alpha\gamma\|\partial_x u\|^2 - 2\alpha\gamma\partial_x u(0)CX \\ \text{and} \quad \beta\frac{d}{dt}\int_0^1 |\partial_x u(x)|^2\, dx &= 2\beta\int_0^1 \partial_x u(x)\partial_{tx}u(x)\, dx \\ &= -2\beta\int_0^1 \partial_{xx}u(x)\partial_t u(x)\, dx + 2\beta\left[\partial_t u\partial_x u\right]_0^1 \\ &= -2\beta\gamma\int_0^1 |\partial_{xx}u(x)|^2\, dx - 2\beta\partial_t u(0)\partial_x u(0) \\ &= -2\beta\gamma\|\partial_{xx}u\|^2 - 2\beta\partial_x u(0)C(AX + Bu(1)). \end{aligned}$$

Merging these three expressions, we can write

$$\frac{d}{dt}V(X,u) = \xi^\top\Psi_1(\gamma,A,B)\xi - 2\alpha\gamma\|\partial_x u\|^2 - 2\beta\gamma\|\partial_{xx}u\|^2$$

where ξ is given by (9) and $\Psi_1(\gamma,A,B)$ is precisely defined later in (13).

The point we want to make is that we will need to estimate both $\|\partial_x u\|^2$ and $\|\partial_{xx}u\|^2$ from below in order to get an estimate from above of $\frac{d}{dt}V(X,u)$. Moreover, the idea is to focus on obtaining this estimation by a quadratic function of the augmented vector ξ. To this end, the use of the Bessel inequality, recalled below, is crucial.

Lemma 1. (Bessel Inequality). Let $y \in L^2(0,1)$, and let $\{e_k \in L^2(0,1), k \in \mathbb{N}\}$ be an orthogonal family for the classical inner product of $L^2(0,1)$. The following integral inequality holds for all $N \in \mathbb{N}$:

$$||y|| \geq \sum_{k=0}^{N} \langle y, e_k\rangle^2 \|e_k\|^{-2}, \quad (10)$$

where, for all $k = 0,\ldots,N$, $\langle y, e_k\rangle = \int_0^1 y(x)e_k(x)\, dx$.

Since we are working here with the heat equation that involves only a second order derivative in space for a first order derivative in time, we will only need to apply this estimate to $\partial_x u$ and to $\partial_{xx}u$. Moreover, since our candidate Lyapunov functional is built using the mean value $\bar{u}$ of $u(x)$, choosing the family $\{e_0, e_1, e_2\}$ as polynomials up to the order two will be enough for our purpose. Let us make this comment more clear by the following statement.

Lemma 2. If (X,u) is the solution of system (1), then the following inequalities hold :

$$(i) \quad ||u||^2 \geq \bar{u}^2,$$

$$(ii) \quad ||\partial_x u||^2 \geq (CX - u(1))^2 + 3(CX + u(1) - 2\bar{u})^2,$$

$$(iii) \quad ||\partial_{xx} u||^2 \geq \partial_x u(0)^2 + 3(2CX - 2u(1) + \partial_x u(0))^2 + 5(6CX + 6u(1) + \partial_x u(0) - 12\bar{u})^2.$$

Proof : The canonical family of monomials $\{1, x, x^2\}$ is orthonormalized for the $L^2(0,1)$-norm into the family $\{e_0(x) = 1, e_1(x) = \sqrt{3}(2x-1), e_2(x) = \sqrt{5}(6x^2-6x+1)\}$. Since the solution (X,u) of system (1) is such that

$$u \in \mathcal{C}(\mathbb{R}_+; H^1(0,1)) \cap L^2(0,+\infty; H^2(0,1)),$$

the following estimations stem from the previous Bessel inequality.

(i)- Estimate (10) for $N = 0$ applied to $y = u(t)$ yields

$$||u||^2 \geq \langle u, e_0 \rangle^2 = \left(\int_0^1 u(x).1\, dx \right)^2 = \bar{u}^2.$$

Of course this inequality is well-known. It is however notable to see that it is also captured by the Bessel inequality, offering a direct method for future extensions.

(ii)- Using (10) at the order $N = 1$ applied to $y = \partial_x u(t)$, boundary conditions in (1) and integrations by parts give

$$\begin{aligned} ||\partial_x u||^2 &\geq \langle \partial_x u, e_0 \rangle^2 + \langle \partial_x u, e_1 \rangle^2, \\ &= (CX - u(1))^2 + 3(CX + u(1) - 2\bar{u})^2. \end{aligned}$$

(iii)- Similarly, estimate (10) at the order 2 applied to $\partial_{xx} u$ yields, thanks to several integrations by parts

$$\begin{aligned} ||\partial_{xx} u||^2 &\geq \langle \partial_{xx} u, e_0 \rangle^2 + \langle \partial_{xx} u, e_1 \rangle^2 + \langle \partial_{xx} u, e_2 \rangle^2 \\ &\geq \partial_x u(0)^2 + 3(2CX - 2u(1) + \partial_x u(0))^2 \\ &\quad + 5(6CX + 6u(1) + \partial_x u(0) - 12\bar{u})^2, \end{aligned}$$

that ends the proof of Lemma 2. □

3. EXPONENTIAL STABILITY RESULT

We provide here a stability result for the coupled system (1), which is based on the proposed Lyapunov functional (6) and the use of Lemma 2.

Theorem 1. Consider system (1) with a given thermal diffusivity $\gamma > 0$ such that the matrix $A + BC$ is non singular. If there exist $\delta > 0$, $\alpha, \beta > 0$, $P \in \mathbb{S}_+^n$, $Q \in \mathbb{R}^n$ and $T \in \mathbb{R}$ satisfying the following two LMIs

$$\Phi = \begin{bmatrix} P & Q \\ Q^\top & T + \alpha \end{bmatrix} \succ 0, \tag{11}$$

$$\Psi(\gamma, A, B) = \Psi_1(\gamma, A, B) - 2\alpha\gamma\Psi_2 - 2\beta\gamma\Psi_3 \prec 0, \tag{12}$$

where

$$\Psi_1(\gamma, A, B) = \begin{bmatrix} \text{He}(PA) & PB & \psi_{13} & A^\top Q \\ * & 0 & -\beta B^\top C^\top & B^\top Q \\ * & * & 0 & -\gamma T \\ * & * & * & 0 \end{bmatrix} \tag{13}$$

$$\Psi_2 = \begin{bmatrix} C & -1 & 0 & 0 \\ C & 1 & 0 & -2 \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} C & -1 & 0 & 0 \\ C & 1 & 0 & -2 \end{bmatrix}$$

$$\Psi_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2C & -2 & 1 & 0 \\ 6C & 6 & 1 & -12 \end{bmatrix}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2C & -2 & 1 & 0 \\ 6C & 6 & 1 & -12 \end{bmatrix}$$

with $\psi_{13} = -\gamma Q - \alpha\gamma C^\top - \beta A^\top C^\top$

then the coupled infinite dimensional system (1) is exponentially stable. Indeed, there exist constants $K > 0$ and $\delta > 0$ such that:

$$E(X,u) \leq K e^{-\delta t} \left(|X^0|_n^2 + \|u^0\|_{H^1(0,1)}^2 \right), \forall t > 0.$$

Proof : The proof consists in showing that if the LMI conditions (11) and (12) are verified for a given $N \geq 0$ then there exist three positive scalars $\varepsilon_1, \varepsilon_2$ and ε_3 such that for all $t > 0$, inequalities (7) and (8) hold.

Existence of ε_1: The computation of $\frac{d}{dt}V$ was already done and yields:

$$\tfrac{d}{dt}V(X,u) = \xi^\top \Psi_1(\gamma, A, B)\xi - 2\alpha\gamma\|\partial_x u\|^2 - 2\beta\gamma\|\partial_{xx} u\|^2$$

where $\Psi_1(\gamma, A, B)$ is defined in (13) and ξ in (9). Using the definition of $\Psi(\gamma, A, B)$ in (12), the derivative of the functional V along the trajectories of the system can be rewritten as

$$\begin{aligned} \frac{d}{dt}V(X,u) = \xi^\top \Psi(\gamma, A, B)\xi &- 2\alpha\gamma\left(\|\partial_x u(t)\|^2 - \xi^\top \Psi_2 \xi\right) \\ &- 2\beta\gamma\left(\|\partial_{xx} u(t)\|^2 - \xi^\top \Psi_3 \xi\right) \end{aligned}$$

Since we assume LMI (12) that writes $\Psi(\gamma) \prec 0$ and $\alpha > 0$, there exists a sufficiently small $\varepsilon_1 > 0$ such that

$$\Psi(\gamma, A, B) \prec -\varepsilon_1 \left(\begin{bmatrix} I_n & 0 & 0 & 0 \\ * & 2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} + 3\Psi_2 \right) \text{ and } \varepsilon_1 \leq \frac{2}{3}\alpha\gamma.$$

Hence, replacing $\Psi(\gamma, A, B)$ by its upper bound leads to

$$\begin{aligned} \frac{d}{dt}V(X,u) &\leq -\varepsilon_1 \xi^\top \begin{bmatrix} I_n & 0 & 0 & 0 \\ * & 2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \xi - 3\varepsilon_1 \xi^\top \Psi_2 \xi \\ &- 2\alpha\gamma\left(\|\partial_x u\|^2 - \xi^\top \Psi_2 \xi\right) - 2\beta\gamma\left(\|\partial_{xx} u\|^2 - \xi^\top \Psi_3 \xi\right) \end{aligned}$$

or equivalently

$$\begin{aligned} &\frac{d}{dt}V(X,u) \leq -\varepsilon_1\left(|X|^2 + 2|u(1)|^2 + 3\|\partial_x u\|^2\right) \\ &-(2\alpha\gamma - 3\varepsilon_1)\left(\|\partial_x u\|^2 - \xi^\top \Psi_2 \xi\right) - 2\beta\gamma\left(\|\partial_{xx} u\|^2 - \xi^\top \Psi_3 \xi\right). \end{aligned}$$

Moreover, one can easily prove that for any $u \in H^1(0,1)$,

$$\|u\|^2 \leq 2|u(1)|^2 + 2\|\partial_x u\|^2,$$

and we get

$$\begin{aligned} |X(t)|^2 + 2|u(1,t)|^2 + 3\|\partial_x u(t)\|^2 &\geq \\ |X|^2 + \|u\|^2 + \|\partial_x u\|^2 &= E(X,u) \end{aligned}$$

so that we can write

$$\begin{aligned} &\frac{d}{dt}V(X,u) \leq -\varepsilon_1 E(X,u) \\ &-(2\alpha\gamma - 3\varepsilon_1)\left(\|\partial_x u\|^2 - \xi^\top \Psi_2 \xi\right) - 2\beta\gamma\left(\|\partial_{xx} u\|^2 - \xi^\top \Psi_3 \xi\right) \end{aligned}$$

In order to conclude the proof of (7), it remains to prove that the two last terms are negative definite. Since ε_1 has been selected so that $2\alpha\gamma - 3\varepsilon_1 > 0$ and since β and γ are positive, one only needs to ensure that

$$\|\partial_x u\|^2 - \xi^\top \Psi_2 \xi \geq 0 \quad \text{and} \quad \|\partial_{xx} u\|^2 - \xi^\top \Psi_3 \xi \geq 0.$$

It is actually a consequence of Lemma 2, since the computation of $\xi^\top \Psi_2 \xi$ and $\xi^\top \Psi_3 \xi$ leads exactly to the right hand side of inequalities (ii) and (iii) of Lemma 2, respectively.

Finally, this allows us to write $\frac{d}{dt}V(X,u) \le -\varepsilon_1 E(X,u)$ which is precisely (7).

Existence of ε_2: Since $\alpha > 0$ and $\Phi \succ 0$ as in LMI (11), there exists a sufficiently small $\varepsilon_2 > 0$ such that

$$\varepsilon_2 \le min(\alpha,\beta) \text{ and } \Phi = \begin{bmatrix} P & Q \\ Q^\top & T+\alpha \end{bmatrix} \succ \varepsilon_2 \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, using also the obvious inequality (i) of Lemma 2, one can then bound $V(X,u)$ from below as follows:

$$\begin{aligned} V(X,u) &= \begin{bmatrix} X \\ \bar{u} \end{bmatrix}^\top \Phi \begin{bmatrix} X \\ \bar{u} \end{bmatrix} + \varepsilon_2(||u||^2 - \bar{u}^2) \\ &\quad +(\alpha-\varepsilon_2)(||u||^2 - \bar{u}^2) + \beta||\partial_x u||^2 \\ &\ge \begin{bmatrix} X \\ \bar{u} \end{bmatrix}^\top \Phi \begin{bmatrix} X \\ \bar{u} \end{bmatrix} + \varepsilon_2(||u||^2 - \bar{u}^2) + \beta||\partial_x u||^2 \\ &\ge \varepsilon_2(|X|_n^2 + \|u\|^2 + ||\partial_x u||^2) = \varepsilon_2 E(X,u). \end{aligned}$$

Existence of ε_3: There exists a sufficiently large scalar $\lambda > 0$ such that $\begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \preceq \lambda \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, defining $\varepsilon_3 = \max\{\lambda+\alpha, \beta\}$ and using again (i) from Lemma 2, we obtain finally

$$V(X,u) \le \lambda |X|_n^2 + \lambda \bar{u}^2 + \alpha\|u\|^2 + \beta\|\partial_x u\|^2 \le \varepsilon_3 E(X,u).$$

As a conclusion, from the solvability of LMIs (11)-(12), we prove the existence of $\varepsilon_1, \varepsilon_2$ and ε_3 satisfying (7)-(8), and get finally the exponential stability of system (1). □

4. ROBUST EXPONENTIAL STABILITY RESULT

Among the potentialities of the stability analysis of the coupled system (1) using an LMI formulation, it is possible to extend Theorem 1 to robust exponential stability. This means that the previous analysis can also address the case of uncertain systems given by system (1) where the thermal diffusivity is not fixed but rather satisfy, for two positive real bounds $\gamma_1 < \gamma_2$, the assumption $\gamma \in [\gamma_1, \gamma_2]$.

Theorem 2. Consider system (1) with an uncertain thermal diffusivity $\gamma \in [\gamma_1, \gamma_2]$. If there exist $\delta > 0$, $\alpha, \beta > 0$, $P \in \mathbb{S}_n$, $Q \in \mathbb{R}^n$ and $T \in \mathbb{R}$ satisfying $\Phi \succ 0$ as in (11) and

$$\Psi(\gamma_j, A, B) \prec 0, \quad j = 1, 2, \tag{14}$$

where the matrices $\Psi(\gamma_j, A, B)$ are defined in (12). Then the uncertain coupled infinite dimensional system (1) is exponentially stable.

Proof : The matrix $\Psi(\gamma, A, B)$ is affine with respect to γ and consequently convex with respect to its first argument.

Therefore if the matrices $\Psi(\gamma_j, A, B)$ are negative definite for all $j = 1, 2$, the matrix $\Psi(\gamma, A, B)$ is also negative definite. Exponential stability follows from Theorem 1. □

5. NUMERICAL EXAMPLES

Example 1. Consider system (1) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -0.1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top.$$

We first note that the matrix $A+BC = \left[\begin{smallmatrix} 0 & 1 \\ -2+b & -0.1 \end{smallmatrix}\right]$ is non singular for any $b \ne 2$. Therefore, Proposition 1 ensures that system (1) with $b \ne 2$ has a unique equilibrium $(0,0)$. Moreover, we note that the matrices A and $A+BC$ have eigenvalues in the left hand side of the complex plane for any $b < 2$. This means that the open loop system $\dot{X} = AX$ and the system controlled by a static output feedback $u(t) = CX(t)$, $\dot{X} = (A+BC)X$ are stable. However, we will show that these two conditions are not sufficient to guarantee the stability of the coupled system (1).

Figure 1 shows the relation between the minimum admissible diffusion parameter γ and the parameter b resulting from the conditions of Theorem 1. One may first note that for $b = 0$, we have $A+BC = A$, meaning that there is no coupling between the ODE and the PDE. Since the matrix A is Hurwitz and since the heat equation is inherently stable, the conditions of Theorem 1 is able to guarantee the stability of both the ODE and the heat PDE separately. When $b = 2$, the matrix $A+BC$ is not invertible and the conditions of Theorem 1 become infeasible. These conditions are also not feasible for $b > 2$. This can be understood by the fact that the matrix $A+BC$ has unstable eigenvalues.

Figure 1 also shows two linear relations between b and the diffusion parameter γ, for $b \in (-\infty, 0)$ and $b \in (0, 2)$. The interpretation of this behavior needs to be accurately investigated in future work.

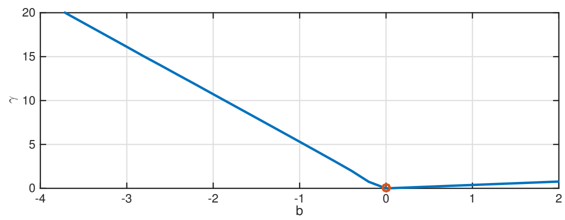


Fig. 1. Evolutions of the minimal admissible diffusion gain γ with respect to the parameter b according to Theorem 1.

Example 2. Consider now the uncertain system (1) with

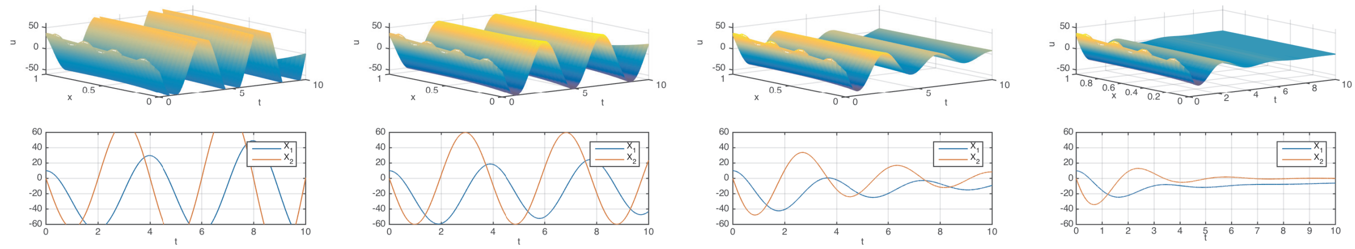
$$A = \begin{bmatrix} -0.1 & 1 \\ 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \; C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^\top \tag{15}$$

We first note that the matrix $A+BC = \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ is non singular, so that Proposition 1 ensures that system (1) has a unique equilibrium $(0,0)$. Moreover, the open-loop system (i.e. the system driven by $\dot{X} = AX$) is unstable.

The conditions of Theorem 1 allow us to guarantee stability of the coupled system (1) for any γ greater than $\gamma_{min} = 1.232$. This result can be interpreted through the idea that the diffusion coefficient γ should be sufficiently large, so that the disturbance due to the boundary coupling with the heat equation can be neglected.

Figure 2 shows four simulations of system (1) with (15) and with four different values of γ with the initial condition $u^0(x) = CX^0 + p_0 x(x-2) + p_1(1 - cos(\omega_1 x))$ and the numerical values $X^0 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, $p_0 = -20$, $p_1 = 10$, $\omega_1 = 8\pi$. Note that this selection of initial condition is compatible with the requirements $u^0(0) = CX^0$ and $\partial_x u^0(1) = 0$.

Remark 2. Simulations of the coupled ODEs-PDE have been performed using classical tools available in the lit-



(a) $\gamma = 1.0$. (b) $\gamma = 1.048$. (c) $\gamma = 1.232$. (d) $\gamma = 1.5$.

Fig. 2. Simulations of the state (X, u) of the coupled system (1) with (15) for four values of the diffusion parameter γ.

erature. The ODE has been discretized using a Runge-Kutta algorithm of order 4 with a principal step δ_t. The PDE have been simulated by performing a backward in time central order difference in space with a step δ_x, with $\delta_t = \delta_x^2/(2\gamma)$ and $\delta_x = 1/100$ to ensure the numerical stability of the approximation.

It can be seen in Figures 2.a and 2.b that the system is unstable for $\gamma = 1$ and 1.048. More particularly, for $\gamma = 1.048$, the solution of the system is oscillating, allowing us to think that it is the minimal value of γ such that system (1) with (15) remains exponentially stable. Figures 2.c and 2.d depict simulations results for larger values of γ for which the solutions are indeed exponentially stable, as expected from the conditions of Theorem 1. Therefore, system (1) is stable for values of γ lower than the minimal one provided by Theorem 1. This remark demonstrates that the stability conditions of Theorem 1 still contain some degree of conservatism. It also reminds us the inherent and classical conservatism of stability conditions addressing stability of time-delay systems (i.e. coupled ODE with a transport PDE) as in Richard (2003); Kharitonov (2012); Niculescu (2001). It has been demonstrated that the Bessel inequality, which was also considered in Seuret and Gouaisbaut (2015) for time-delay systems, represents a remarkable and efficient tool in the reduction of the conservatism. Therefore, a direction of future research would consist in reducing the conservatism of the stability conditions following the principles provided in these papers.

6. DISCUSSION AND CONCLUSION

In this paper, we have provided a novel approach to assess stability of coupled ODE-Heat PDE systems. The method relies on an efficient construction of dedicated Lyapunov functionals allowing to derive diffusion parameter-dependent stability conditions. These tractable conditions are expressed in terms of LMIs and are obtained thanks to the application of integral inequalities resulting from the more general formulation of the Bessel inequality. This work can be seen as a first and preliminary contribution to the fields of infinite dimensional systems. Future research will include the reduction of the conservatism in the LMI conditions and the consideration of more complex PDE in the coupling with ODEs.

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