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# Allowable delay sets for the stability analysis of linear time-varying delay systems using a delay-dependent reciprocally convex lemma <sup>\*</sup>

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**Abstract:** This paper addresses the stability analysis of linear systems subject to a time-varying delay. The contribution of this paper is twofolds. First, we aim at presenting a new matrix inequality, which can be seen as an improved version of the reciprocally convex combination, which provides a more accurate delay-dependent lower bound. When gathering this new inequality with the Wirtinger-based integral inequality, efficient stability conditions expressed in terms of LMI are designed and show a clear reduction of the conservatism with a reasonable associated computational cost. The second original contribution of this paper consists in noting that stability conditions issued from the Wirtinger-based integral inequality depends in an affine manner on the bounds of the delay function and also on its derivative. This allows to refine the definition of allowable delay set and to relax usual convex on the delay function. As a result of this new characterization, the LMI conditions allows obtaining stability regions for slow time-varying delay systems which are very closed to the constant delay case.

*Keywords:* Time-delay systems, integral inequalities, matrix inequality, reciprocally convex lemma.

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## 1. INTRODUCTION

This paper aims at providing less conservatism and computationally efficient stability conditions for linear systems subject to fast-varying delays. This topic of research has attracted many researchers over the past decades. The main difficulties for the study of such a class of systems rely on two technical steps that are the derivation of efficient integral and matrix inequalities. Indeed, the differentiation of usual candidates for being Lyapunov-Krasovskii functionals leads to integral quadratic terms that cannot be included straightforwardly in a linear matrix inequality (LMI) setup. Including these terms requires the use of integral inequalities such as Jensen (see for instance Gu (2000)), Wirtinger-based provided in Seuret and Gouaisbaut (2013), auxiliary-based from Park et al. (2015) or Bessel inequalities developed in Seuret and Gouaisbaut (2015). Although these inequalities have shown a great interest for constant delay systems, their application to time- or fast-varying delays leads to additional difficulties related to the non convexity of the resulting terms. Then, some matrix inequalities are employed to derive convex conditions. The first method corresponds to the application of Young's inequality or Moon's inequality, which basically results from the positivity of a square positive definite term. It can also be noted that the recent free-matrix inequality from Zeng et al. (2015) can be interpreted as the merge of the Wirtinger-based inequality and Moon's inequality. Recently, the reciprocally convex lemma was proposed in Park et al. (2011). The novelty

of this method consists to gather the non convex terms into a single expression to derive an accurate convex inequality. It was notably shown that the conservatism of the reciprocally convex lemma from Park et al. (2011) and the Moon's inequality are similar when considering Jensen-based stability criteria, with a lower computational burden.

In the present paper, the objective is to refine the reciprocally convex lemma by introducing delay dependent terms. The resulting lemma includes the initial reciprocally convex lemma as a particular case, and examples show a clear reduction of conservatism with respect to the literature at a reasonable increase of the computational cost. This lemma is also in the same vein as the recent contribution on the relaxation of the reciprocally convex lemma provided in Zhang et al. (2016). In this paper, a relevant lemma has been provided since it does not require the introduction of additional decision variables. In the present paper, a new technical lemma is provided and introduces new slack variables in the reciprocally convex lemma in order to reduce the conservatism. This lemma is then employed to derive a new stability theorem for linear systems subject to a time-varying delay, expressed in terms of LMIs, which explicitly depends on the bounds of the delay function and of its derivative. Two cases are then considered. The first one refers to the usual constraints where the delay function and its time-derivative are considered independently. The second case corresponds to a refined and new characterization of the allowable delay set, which leads to the notable improvements on this example. In particular, we show, on the examples, that the results obtained for slow-varying delays are equal to the results obtained for constant delay.

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**Notations:** Throughout the paper  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space and  $\mathbb{R}^{n \times m}$  and  $\mathbb{S}^n$  are the set of  $n \times m$  real matrices and of  $n \times n$  real symmetric matrices, respectively. For any  $P \in \mathbb{S}^n$ ,  $P \succ 0$  means that  $P$  is symmetric positive definite. For any matrices  $A, B, C$  of appropriate dimension, the matrix  $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$  stands for  $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ . The matrices  $I_n$  and  $0_{n,m}$  represent the identity and null matrices of appropriate dimension and, when no confusion is possible, the subscript will be omitted. For any  $h > 0$  and any function  $x : [-h, +\infty) \rightarrow \mathbb{R}^n$ , the notation  $x_t(\theta)$  stands for  $x(t + \theta)$ , for all  $t \geq 0$  and all  $\theta \in [-h, 0]$ .

## 2. PROBLEM FORMULATION AND PRELIMINARIES

### 2.1 System data

Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_2, 0], \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi$  is the initial condition and  $A, A_d \in \mathbb{R}^{n \times n}$  are constant matrices. There exist positive scalars  $h_2 \geq 0$  and  $d_1 \leq d_2 \leq 1$  such that

$$\begin{cases} h(t) \in [0, h_2], & \forall t \geq 0, \\ \dot{h}(t) \in [d_1, d_2], & \forall t \geq 0, \end{cases} \quad (2)$$

When possible, the time argument of the delay functions  $h(t)$  and  $\dot{h}(t)$  will be omitted.

Providing efficient stability conditions for time-varying or fast-varying delay systems relies on the accuracy of matrix or integral inequalities under consideration. On the one hand, much attention has been paid recently to integral inequalities, as mentioned in the introduction. On the other hand, when considering time-varying delays, these inequalities have to be combined with matrix inequalities such as Young's inequality Moon et al. (2001) or the reciprocally convex lemma Park et al. (2011) to remove the non convex term. In this paper we present a new matrix inequality aiming at reducing the conservatism of the reciprocally convex lemma. To show this reduction, we will first concentrate, in this paper, on the Wirtinger-based inequality recalled in the next lemma. Nevertheless, the main result of this paper can be adapted to other integral inequalities.

*Lemma 1.* Let  $R \succ 0$  be in  $\mathbb{S}_+^n$  and  $x$  be a continuously differentiable function from  $[a, b]$  (with  $a < b$ ) to  $\mathbb{R}^n$ . The following inequality holds

$$(b - a) \int_a^b \dot{x}^T(s) R \dot{x}(s) ds \geq \omega_0^T R \omega_0^T + 3\omega_1^T R \omega_1^T,$$

where

$$\omega_0 = x(a) - x(b), \quad \omega_1 = x(a) + x(b) - \frac{2}{b-a} \int_a^b x(s) ds.$$

## 3. EXTENDED RECIPROCALLY CONVEX INEQUALITY

This section is devoted to the derivation of new matrix inequalities which refines the celebrated reciprocally convex combination lemma from Park et al. (2011). A general inequality is first provided and then a numerically

tractable corollary are provided. An extended version of the reciprocally convex combination lemma is provided below.

*Lemma 2.* Let  $R$  be a positive definite matrix in  $\mathbb{S}^n$  for a given integer  $n > 0$ . If there exist two matrices  $X_1, X_2$  in  $\mathbb{S}^n$  and  $Y_1, Y_2$  in  $\mathbb{R}^{n \times n}$  such that

$$\begin{bmatrix} R - X_1 & -Y_1 \\ * & R \end{bmatrix} \succ 0, \quad \begin{bmatrix} R & -Y_2 \\ * & R - X_2 \end{bmatrix} \succ 0, \quad (3)$$

then the following inequality holds for all  $\alpha \in [0, 1]$

$$\begin{bmatrix} \frac{1}{\alpha} R & 0 \\ * & \frac{1}{1-\alpha} R \end{bmatrix} \succeq \begin{bmatrix} R & 0 \\ * & R \end{bmatrix} + \begin{bmatrix} (1-\alpha)X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \alpha X_2 \end{bmatrix}. \quad (4)$$

*Proof :* If inequalities (3) are verified, then a convex combination of these two equations leads to the inequality

$$\begin{bmatrix} R & 0 \\ * & R \end{bmatrix} - \begin{bmatrix} \alpha X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & (1-\alpha)X_2 \end{bmatrix} \succ 0,$$

for all  $\alpha \in [0, 1]$ . Pre- and post-multiplying this inequality by the matrix  $\begin{bmatrix} \beta I & 0 \\ 0 & \beta^{-1} I \end{bmatrix}$ , where  $\beta = \sqrt{\frac{1-\alpha}{\alpha}}$  and  $\alpha \in (0, 1)$ , leads to

$$\begin{bmatrix} \frac{1-\alpha}{\alpha} R & 0 \\ * & \frac{\alpha}{1-\alpha} R \end{bmatrix} - \begin{bmatrix} \frac{\alpha(1-\alpha)}{\alpha} X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \frac{\alpha(1-\alpha)}{1-\alpha} X_2 \end{bmatrix} \succ 0,$$

for all  $\alpha \in (0, 1)$ . Finally noting that  $\frac{1-\alpha}{\alpha} = \frac{1}{\alpha} - 1$  and  $\frac{\alpha}{1-\alpha} = \frac{1}{1-\alpha} - 1$ , the previous inequality can be rewritten as

$$\begin{bmatrix} \frac{1}{\alpha} R & 0 \\ * & \frac{1}{1-\alpha} R \end{bmatrix} - \begin{bmatrix} R & 0 \\ * & R \end{bmatrix} - \begin{bmatrix} (1-\alpha)X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \alpha X_2 \end{bmatrix} \succ 0,$$

which concludes the proof.  $\diamond$

In the previous Lemma, it is easy to see that, selecting  $X_1 = X_2 = 0$  and  $Y_1 = Y_2 = Y$ , inequalities (3) resume to  $\begin{bmatrix} R & -Y \\ * & R \end{bmatrix} \succ 0$  (or equivalently  $\begin{bmatrix} R & Y \\ * & R \end{bmatrix} \succ 0$ ) and the inequality (4) recovers the reciprocally convex combination lemma from Park et al. (2011). Therefore, Lemma 2 authorizes more degrees of freedom in the definition of the lower bound of the matrix  $\mathcal{R}(\alpha)$ , whose efficiency will be demonstrated in the next developments.

## 4. STABILITY ANALYSIS

### 4.1 Main result

Based on the previous developments, the following stability theorem is provided.

*Theorem 1.* Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ ,  $X_1, X_2$  in  $\mathbb{S}^{2n}$  and two matrices  $Y_1, Y_2$  in  $\mathbb{R}^{2n \times 2n}$ , such that the conditions

$$\begin{bmatrix} \tilde{R} - X_1 & Y_1 \\ * & \tilde{R} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R} & Y_2 \\ * & \tilde{R} - X_2 \end{bmatrix} \succeq 0, \quad (5)$$

$$\Phi(0, d_1) \prec 0, \quad \Phi(h_2, d_1) \prec 0, \quad \Phi(0, d_2) \prec 0, \quad \Phi(h_2, d_2) \prec 0, \quad (6)$$

are satisfied where

$$\begin{aligned}
\Phi(\theta, \eta) &= \Phi_0(\theta, \eta) - G_2^T \Psi(\theta) G_2 \\
\Phi_0(\theta, \eta) &= He \{ G_1^T(\theta) P G_0(\eta) \} + \hat{S}(\eta) + h_2^2 g_0^T R g_0, \\
\hat{S}(\eta) &= \text{diag}(S_1, (1-\eta)(S_2 - S_1), -S_2, 0_{2n}), \\
\tilde{R} &= \text{diag}(R, 3R), \\
\Psi(\theta) &= \begin{bmatrix} \tilde{R} + \frac{h_2 - \theta}{h_2} X_1 & \frac{\theta}{h_2} Y_1 + \frac{h_2 - \theta}{h_2} Y_2 \\ * & \tilde{R} + \frac{\theta}{h_2} X_2 \end{bmatrix},
\end{aligned} \tag{7}$$

and where

$$\begin{aligned}
g_0 &= [A \ A_d \ 0 \ 0 \ 0], \\
G_0(\eta) &= \begin{bmatrix} A & A_d & 0 & 0 & 0 \\ I & -(1-\eta)I & 0 & 0 & 0 \\ 0 & (1-\eta)I & -I & 0 & 0 \end{bmatrix}, \\
G_1(\theta) &= \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta I & 0 \\ 0 & 0 & 0 & 0 & (h_2 - \theta)I \end{bmatrix}, \\
G_2 &= \begin{bmatrix} I & -I & 0 & 0 & 0 \\ I & I & 0 & -2I & 0 \\ 0 & I & -I & 0 & 0 \\ 0 & I & I & 0 & -2I \end{bmatrix}.
\end{aligned} \tag{8}$$

Then, system (1) is asymptotically stable for all time-varying delay  $h$  satisfying (2).

*Proof:* Consider the same Lyapunov-Krasovskii functional as in Seuret and Gouaisbaut (2013), given by

$$\begin{aligned}
V(x_t, \dot{x}_t) &= \begin{bmatrix} \int_{t-h(t)}^t x(s) ds \\ \int_{t-h_2}^{t-h(t)} x(s) ds \end{bmatrix}^T P \begin{bmatrix} \int_{t-h(t)}^t x(s) ds \\ \int_{t-h_2}^{t-h(t)} x(s) ds \end{bmatrix} \\
&+ \int_{t-h(t)}^t x^T(s) S_1 x(s) ds + \int_{t-h_2}^{t-h(t)} x^T(s) S_2 x(s) ds \\
&+ h_2 \int_{-h_2}^0 \int_{t+\theta}^t \dot{x}^T(s) R \dot{x}(s) ds.
\end{aligned} \tag{9}$$

This functional is positive definite since the matrices  $P$ ,  $S_1$ ,  $S_2$  and  $R$  are symmetric positive definite. Note that it would be also possible to include more terms such as, for instance, triple integral terms. However, we want to show in this paper the reduction of the conservatism related to the use of Lemma 2 compared to the reciprocally convex combination lemma.

The derivative of the functionals along the trajectories of the system leads to

$$\dot{V}(x_t, \dot{x}_t) = \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t, \dot{x}_t). \tag{10}$$

The next developments consist in providing an upper bound of  $\dot{V}$ , expressed using the augmented vector  $\zeta(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$  where

$$\zeta_1(t) = \begin{bmatrix} x(t) \\ x(t-h) \\ x(t-h_2) \end{bmatrix}, \quad \zeta_2(t) = \begin{bmatrix} \frac{1}{h} \int_{t-h}^t x^T(s) ds \\ \frac{1}{h_2-h} \int_{t-h_2}^{t-h} x^T(s) ds \end{bmatrix}.$$

To do so, we first note

$$\begin{bmatrix} x(t) \\ \int_{t-h}^t x(s) ds \\ \int_{t-h_2}^{t-h} x(s) ds \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & hI \\ 0 & 0 & (h_2-h)I \end{bmatrix} \zeta(t) = G_1(h) \zeta(t).$$

It is also easy to see that

$$\begin{aligned}
\frac{d}{dt} \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s) ds \\ \int_{t-h_2}^{t-h} x(s) ds \end{bmatrix} &= \begin{bmatrix} \dot{x}(t) \\ x(t) - (1-\dot{h})(t-h) \\ (1-\dot{h})x(t-h) - (t-h_2) \end{bmatrix} \\
&= G_0(\dot{h}) \zeta(t).
\end{aligned}$$

Hence, the derivative of  $V_1$  along the trajectories of the system leads to

$$\dot{V}_1(x_t) = \zeta^T(t) \left( G_1^T(h) P G_0(\dot{h}) + G_0^T(\dot{h}) P G_1(h) \right) \zeta(t).$$

According to the definition of the matrix  $\tilde{S}$  given in (8), differentiating  $V_2$  yields

$$\dot{V}_2(x_t) = \zeta^T(t) \tilde{S}(\dot{h}) \zeta(t).$$

The derivative of the last term  $V_3$  leads to

$$\dot{V}_3(x_t, \dot{x}_t) = h_2^2 \dot{x}^T(t) R \dot{x}(t) - h_2 \int_{t-h_2}^t \dot{x}^T(s) R \dot{x}(s) ds.$$

Noting that  $\dot{x}(t) = g_0 \zeta(t)$ , where  $g_0$  is given in (8), the previous expression can be rewritten as follows

$$\dot{V}(x_t, \dot{x}_t) = \zeta^T(t) \Phi_0(h, \dot{h}) \zeta(t) - h_2 \int_{t-h_2}^t \dot{x}^T(s) R_2 \dot{x}(s) ds.$$

where  $\Phi_0(h, \dot{h})$  is given in (7). Applying Lemma 1 to the integral term, after splitting the second integral into two parts, leads to

$$\dot{V}(x_t, \dot{x}_t) \leq \zeta^T(t) \left( \Phi_0(h, \dot{h}) - G_2^T \begin{bmatrix} \frac{h_2}{h} \tilde{R}_2 & 0 \\ * & \frac{h_2}{h_2-h} \tilde{R}_2 \end{bmatrix} G_2 \right) \zeta(t).$$

where  $G_2$  is given in (8). Applying Corollary 2, if there exist matrices  $X_1, X_2$  in  $\mathbb{S}^{2n}$  and  $Y_1, Y_2$  in  $\mathbb{R}^{2n \times 2n}$  such that conditions (5) hold, then the following inequality holds

$$\dot{V}(x_t, \dot{x}_t) \leq \zeta^T(t) \Phi(h, \dot{h}) \zeta(t). \tag{11}$$

where  $\Phi(h, \dot{h})$  is given in (6). Therefore the system (1) is asymptotically stable if the LMI  $\Phi(h, \dot{h}) \prec 0$  is satisfied for all  $h \in [0, h_2]$  and  $\dot{h} \in [d_1, d_2]$ . Since  $\Phi(h, \dot{h})$  is affine with respect to both  $h$  and  $\dot{h}$ , a necessary and sufficient condition is to test the LMI only on the vertices of the intervals, leading to conditions (6). To conclude, if these two conditions hold, the system (1) is asymptotically stable for all time-varying delay functions satisfying (2).  $\diamond$

It is worth noting that the proof of Theorem 1 is very similar to the one provided in Seuret et al. (2013). The only difference relies on the use of Lemma 2. The impact in terms of reduction of the conservatism will be exposed in the Example Section.

#### 4.2 Reduction of the number of decision variables

In the previous theorem, the number of decision variables can be reduced by introducing some constraints on the

slack variables introduced by application of Lemma 2. This relaxation is proposed in the following corollary.

*Corollary 1.* Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ ,  $X$  in  $\mathbb{S}^{2n}$  and a matrix  $Y$  in  $\mathbb{R}^{2n \times 2n}$ , such that conditions (5) and (6) are verified with

$$X_1 = X_2 = X \text{ and } Y_1 = Y_2 = Y.$$

Then system (1) is asymptotically stable for all time-varying delay  $h$  satisfying (2).

*Remark 1.* The reduction of the computational complexity of the resulting stability conditions leads obviously to an increase of the conservatism of the stability conditions, as it will be showed in the example section. This shows again the traditional tradeoff between computational complexity and conservatism.

#### 4.3 Stability result based on Moon's inequality

The success of the reciprocally convex combination lemma over Moon's inequality relies on the fact that when employing it for a stability theorem based on the Jensen inequality, equivalent results were obtained with a significantly reduced number of decision variables. In the following paragraph we will present a similar result to Theorem 1, which is based on the application of Moon's inequality instead of Lemma 2. This leads to the following result:

*Theorem 2.* Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ , and a matrix  $Y$  in  $\mathbb{R}^{5n \times 4n}$ , such that the conditions

$$\bar{\Phi}(0, d_1) \prec 0, \bar{\Phi}(h_2, d_1) \prec 0, \bar{\Phi}(0, d_2) \prec 0, \bar{\Phi}(h_2, d_2) \prec 0, \quad (12)$$

are satisfied where

$$\bar{\Phi}(\theta, \eta) = \begin{bmatrix} \Phi_0(\theta, \eta) - He\{[Y_1 Y_2] G_2\} & \frac{\theta}{h_2} Y_1 & \frac{h_2 - \theta}{h_2} Y_2 \\ * & \frac{\theta}{h_2} \tilde{R} & 0 \\ * & * & \frac{h_2 - \theta}{h_2} \tilde{R} \end{bmatrix} \quad (13)$$

and where the matrices  $\Phi_0(\theta, \eta)$ ,  $\tilde{R}$  and  $G_2$  are given in (8). Then, system (1) is asymptotically stable for all time-varying delay  $h$  satisfying (2).

*Remark 2.* Recently, a novel contribution based on *Free-Weighting Matrix Inequality* was proposed in Zeng et al. (2015). It has been shown in this paper that an alternative presentation of the Wirtinger-based integral inequality (Lemma 1) can be presented by an efficient introduction of free-weighting-matrices leading to less conservative results compared to the use of the Wirtinger-based inequality. In this paper, we will compare the various results presented here with Corollary 1 of Zeng et al. (2015), which proposes exactly the same Lyapunov-Krasovskii functional as the one presented in (9). This will allow a fair comparison between the various inequalities employed is all these results. Note that the main stability theorem of Zeng et al. (2015) exploits additional terms in the construction of the functional, leading to a reduction of the conservatism. As a by-product of the contribution of Gyurkovics (2015), the reduction of the conservatism is not related to the Wirtinger-based inequality but rather on the use of other

Th.	No. of variables
Ariba and Gouaisbaut (2009)	$480n^2 + 8n$
Fridman and Shaked (2002)	$5.5n^2 + 1.5n$
He et al. (2007)	$3n^2 + 3n$
Park and Ko (2007)	$11.5n^2 + 4.5n$
Seuret and Gouaisbaut (2013)	$10n^2 + 3n$
Zeng et al. (2015)	$54n^2 + 9n$
Zeng et al. (2013)	$17n^2 + 5n$
Zhang et al. (2016) <sup>1</sup> (Th1.C2)	$10n^2 + 3n$
Zhang et al. (2016) <sup>2</sup> (Th1.C2)	$23n^2 + 4n$
Cor. 1	$12n^2 + 4n$
Th. 1	$18n^2 + 5n$
Th. 2	$26n^2 + 3n$

Table 1. Number of decision variables involved in several conditions from the literature and in Theorem 1 and its corollaries

technical bounding lemmas as the reciprocally convex combination lemma or the Moon inequality. We will not present the numerical results in the present paper since our goal is to show the conservatism of integral and matrix inequalities and their associated numerical complexity.

#### 4.4 Illustrative Examples

In this section, we will consider two academic examples taken from the literature. Our goal is to illustrate and compare the efficiency of the conditions presented in Theorems 1 and 2 and Corollary 1 and for various conditions from the literature dedicated to the stability analysis of linear systems with time-varying delays. Before entering into the numerical results, we would like to point out in Table 1, the number of decision variables involved in the conditions presented in this paper and in existing results from the literature. For the two next examples, we expose in Tables 2 and 3 the maximal upper-bound,  $h_2$  of the delay function for various values bounds on the derivative of the delay function, i.e.  $d_2 (= -d_1)$ .

There exists a large number of papers dealing with the stability analysis of such a class of systems. Because of space limitations, we consider only few representative conditions from the literature. Firstly, we have considered the conditions which use Jensen's inequality (Fridman et al. (2009) and Park et al. (2011)), Wirtinger-based inequality (Seuret et al. (2013)), auxiliary-based inequality Park et al. (2015) and the recent free-matrix-based inequality (Zeng et al. (2015)); secondly, we also study conditions that are based on Young/Moon inequality Fridman et al. (2009); Zeng et al. (2015), on the Reciprocally convex combination lemma Park et al. (2011); Seuret et al. (2013); Park et al. (2015) or on the recent relaxed reciprocally convex inequality developed in Zhang et al. (2016). A last comment is proposed on the contribution presented in Theorem 2.C2. from Zhang et al. (2016). Indeed, the conditions proposed in this theorem are based on the auxiliary functions based inequality from Park et al. (2015) or the Bessel-Legendre inequality Seuret and Gouaisbaut (2015) of order 2, which are proven to be less conservative than the Wirtinger-based inequality. Therefore, it is expected that the conditions presented in Theorem 2.C2. from Zhang et al. (2016) are less conservative than the one from Theorem 1. It is also worth noting that despite an increasing number of decision variables, the stability conditions provided in Theorem 1

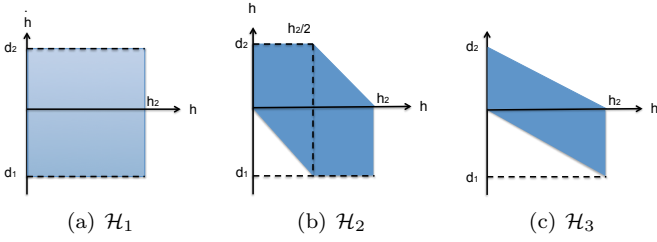


Fig. 1. Graphical illustration of allowable delay sets.

$d_2$	0.0	0.1	0.5	0.8	1
Fridman et al. (2002)	4.47	3.60	2.00	1.36	0.99
He et al. (2007)	4.47	3.60	2.04	1.49	1.34
Park and Ko (2007)	4.47	3.65	2.33	1.93	1.86
Ariba et al. (2009)	6.11	4.79	2.68	1.95	1.60
Zeng et al. (2013) (N=3)	5.92	4.62	2.44	2.07	2.07
Seuret et al.(2013)*	6.05	4.70	2.42	2.13	2.12
Zeng et al. (2015) (Cor.1)*	6.05	4.71	2.45	2.21	2.18
Zeng et al. (2015) (Th.1)	6.05	4.78	3.05	2.61	-
Zhang et al. (2016) <sup>1</sup> *	6.05	4.70	2.42	2.20	2.20
Zhang et al. (2016) <sup>2</sup>	6.16	4.71	2.60	2.37	2.31
Cor. 1*	6.05	4.70	2.42	2.20	2.20
Th. 1*	6.05	4.71	2.48	2.24	2.24
Th. 2*	6.05	4.71	2.48	2.30	2.30

Table 2. Example 1: Admissible upper bound of  $h_2$  for various values of  $d_2 = -d_1$ . The mark ‘\*’ means that the stability conditions are based on the same functional.

$d_2$	0	0.05	0.1	0.5	3
Seuret et al.(2013)*	3.03	2.55	2.36	1.70	1.65
Zhang et al. (2016) <sup>1</sup> *	3.03	2.55	2.37	1.70	1.64
Cor. 1	3.03	2.55	2.37	1.70	1.67
Th. 1	3.03	2.55	2.37	1.72	1.68
Th. 2	3.03	2.55	2.37	1.72	1.70

Table 3. Example 2: Admissible upper bound of  $h_2$  for various values of  $d_2 = -d_1$ .

and in Corollary 1 deliver the same or less conservative results than the similar analysis developed in Zhang et al. (2016).

**Example 1:** Consider the following much-studied linear time-delay system (1) with

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}.$$

The results obtained by solving Theorem 1 and its Corollary show a clear reduction of the conservatism. Moreover, the improvements due to the use of Lemma 2 and its Corollary can be seen when comparing the results obtained with Seuret et al. (2013) and the stability conditions provided in the present paper. Indeed the only difference between these two papers is the use of the delay-dependent reciprocally convex lemma. Moreover, it is worth noting that Theorem 1 and its corollaries provide less conservative results, on this example, than other conditions from the literature except for Park et al. (2015) with  $h_1 = 3$ . This improvement of Park et al. (2015) can be explained by the use of the auxiliary function integral inequality, which is less conservative than the Wirtinger inequality. It is also worth noting that Theorem 1 and its corollaries leads in general to the same results except

for small lower bounds  $h_1 = 0$  even if the computational complexities are different.

**Example 2:** Consider the following example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

For this example, a comparison of Theorem 1, Corollary 1 and Seuret and Gouaisbaud (2013) is presented. The goal is to demonstrate the conservatism introduced by the three matrix inequalities considered in this paper, namely, the original reciprocally convex combination lemma from Park et al. (2011), Lemma 2 and the constrained version of Lemma 2 (i.e., with  $X_1 = X_2$  and  $Y_1 = Y_2$ ). The numerical results shows that for small  $d_2$ , we obtain nearly the same upperbound for the three matrix inequalities and also for Seuret et al.(2013) and Zhang et al. (2016). It indicates that at least on this example, some slack variables are useless. Nevertheless, for bigger value of  $d_2$ , the introduction of slack variables allow a reduction of conservatism as expected.

## 5. REFINED CHARACTERIZATION OF THE ALLOWABLE DELAY SETS

### 5.1 Stability theorem

In the previous analysis, the stability conditions results from the fact that the matrix  $\Phi(h, \dot{h})$ , defined in (7) is affine with respect to the delay  $h$  and, also, its derivative  $\dot{h}$ . The result of Theorem 1 can be interpreted as the satisfaction of the LMI  $\Phi(h, \dot{h}) \prec 0$  for all values of the delay  $h \in [h_1, h_2]$  and of its derivative  $\dot{h} \in [d_1, d_2]$ . This corresponds to the polytope in  $[h \ \dot{h}]^T$  given by

$$\begin{bmatrix} h \\ \dot{h} \end{bmatrix} \in \mathcal{H}_1 = [0, h_2] \times [d_1, d_2] = Co \left\{ \begin{bmatrix} 0 \\ d_1 \end{bmatrix}, \begin{bmatrix} 0 \\ d_2 \end{bmatrix}, \begin{bmatrix} h_2 \\ d_2 \end{bmatrix}, \begin{bmatrix} h_2 \\ d_1 \end{bmatrix} \right\}. \quad (14)$$

and depicted in Figure 1(a). Taking a careful attention at the definition of this set, the boundary points

$$\begin{bmatrix} h \\ \dot{h} \end{bmatrix} = \begin{bmatrix} 0 \\ d_1 \end{bmatrix} \text{ with } d_1 < 0, \quad \begin{bmatrix} h \\ \dot{h} \end{bmatrix} = \begin{bmatrix} h_2 \\ d_2 \end{bmatrix} \text{ with } d_2 > 0,$$

contradict the fact that  $h_1$  and  $h_2$  are respectively the minimum and maximum values of the delay  $h$ . Therefore, one may replace these two boundary points by another boundary points  $(h, \dot{h}) = (h_1, 0)$  (with  $d_1 < 0$ ) and  $(h, \dot{h}) = (h_2, 0)$ , leading to some new allowable delay set given for instance by,

$$\begin{bmatrix} h \\ \dot{h} \end{bmatrix} \in \mathcal{H}_2 = Co \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d_2 \end{bmatrix}, \begin{bmatrix} h_2/2 \\ d_2 \end{bmatrix}, \begin{bmatrix} h_2 \\ 0 \end{bmatrix}, \begin{bmatrix} h_2 \\ d_1 \end{bmatrix}, \begin{bmatrix} h_2/2 \\ d_1 \end{bmatrix} \right\}, \quad (15)$$

or

$$\begin{bmatrix} h \\ \dot{h} \end{bmatrix} \in \mathcal{H}_3 = Co \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d_2 \end{bmatrix}, \begin{bmatrix} h_2 \\ 0 \end{bmatrix}, \begin{bmatrix} h_2 \\ d_1 \end{bmatrix} \right\}, \quad (16)$$

which are depicted in Figure 1(b) and (c). These new definition of the delay sets prevent from the situation to get the delay  $h$  is at its maximum  $h_2$  (or minimum  $h_1$ ) as well as its derivative positive (or negative). This selection reduces notably the size of the polytope. Based on this remark, another corollary of Theorem 1 is provided below

$d_2$	0.0	0.1	0.2	0.5	0.8	1
Example 1						
$\mathcal{H}_1$	6.05	4.73	3.89	2.53	2.27	2.24
$\mathcal{H}_2$	6.05	5.99	5.34	3.43	2.75	2.63
$\mathcal{H}_3$	6.05	6.05	5.99	4.80	3.83	2.65
Example 2						
$\mathcal{H}_1$	3.03	2.37	2.14	1.76	1.67	1.67
$\mathcal{H}_2$	3.03	2.90	2.75	2.31	2.04	1.97
$\mathcal{H}_3$	3.03	2.96	2.89	2.64	2.40	2.31

Table 4. Example 1 & 2: Admissible upper bound of  $h_2$  obtained by Theorem 1 for various values of  $d_2 = -d_1$  and allowable delay sets  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$ .

*Corollary 2.* Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{2n}$ ,  $S_1, S_2, R$  in  $\mathbb{S}_+^n$ ,  $X$  in  $\mathbb{S}_+^{2n}$ , and two matrices  $Y$  in  $\mathbb{R}^{2n \times 2n}$ , such that the conditions (5) and

$$\Phi(h, \dot{h}) \prec 0, \quad \forall \begin{bmatrix} h \\ \dot{h} \end{bmatrix} \in \mathcal{H}_2 \quad (\text{or } \mathcal{H}_3), \quad (17)$$

are satisfied where the matrix  $\Phi$  is given in (7). Then system (1) is asymptotically stable for all time-varying delay  $h$  satisfying

$$\begin{bmatrix} h \\ \dot{h} \end{bmatrix} \in \mathcal{H}_2 \quad \left( \text{or } \begin{bmatrix} h \\ \dot{h} \end{bmatrix} \in \mathcal{H}_3 \right).$$

A natural expected consequence is a reduction of the conservatism of the stability conditions as it will be detailed in the example section.

*Remark 3.* In Figure 1, one can see that, for given values of  $h_2$ ,  $d_1$  and  $d_2$ , the associated sets  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ , and  $\mathcal{H}_3$  verifies

$$\mathcal{H}_3 \subset \mathcal{H}_2 \subset \mathcal{H}_1.$$

This naturally implies some inclusions in the allowable bounds of the delay functions and of its derivative.

### 5.2 Impact of the delay set on illustrative examples

In this section, we will consider the two examples provided in Section 4.4. We propose to illustrate the effect of the allowable delay sets through Table 4. For both examples, one can see that, for fixed values of  $d_2 = -d_1$ , the maximal allowable delay obtained for  $\mathcal{H}_1$  is lower than those obtained for  $\mathcal{H}_2$ , which is again lower than those calculated for  $\mathcal{H}_3$ . The difference between the results obtained for  $\mathcal{H}_1$  and for  $\mathcal{H}_3$  are notably different (around 20%). Another relevant aspect of this new characterization of the allowable delay set is that, for Example 1, the conservatism of the stability conditions remain the same for slow varying delays and for constant delay. This behavior seems indeed natural that slow variation of the delay should not impact drastically the maximum allowable bound of the delay function  $h_2$ .

As a conclusion of this section, one can see that the recent stability conditions developed in the literature which are linearly dependent on  $h$  and  $\dot{h}$  allows considering a wider class of allowable delay sets which take into account more information on the delay function.

## 6. CONCLUSIONS

In this paper, a new reciprocally convex lemma has been provided. The novelty of this technical lemma brings a notable reduction of the conservatism of LMI stability conditions for time-varying delay systems with a reasonable

additional computational burden. In addition we point out a novel idea of allowable delay sets which consists in considering a more accurate definition of the set in which the delay function lies, leading to a significant improvement.

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