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On ISS-Lyapunov functions for infinite-dimensional linear control systems subject to saturations

Swann Marx¹, Yacine Chitour² and Christophe Prieur³

Abstract—This article deals with the derivation of ISS-Lyapunov functions for infinite-dimensional linear systems subject to saturations. Two cases are considered: 1) the saturation acts in the same space as the control space; 2) the saturation acts in another space, especially a Banach space. For the first case, an explicit ISS-Lyapunov function can be derived. For the second case, we can only ensure the existence of an ISS-Lyapunov function.

I. INTRODUCTION

In recent decades, a great deal of effort has been dedicated to the development of tools for the analysis of systems with a saturated control in a finite-dimensional framework (see [28], for a nice introduction to the topic as well as the main techniques used). It is well known, in the finite-dimensional framework, that without any Lyapunov stability property for the open-loop system, the origin cannot be globally asymptotically stable when closing the loop with a saturated feedback-law. Moreover, even if the open-loop system is Lyapunov stable, simply saturating a stabilizing feedback can lead to undesirable behavior for the asymptotic stability of the closed-loop system (see e.g., [6]). In these cases, the feedback-law has to be designed taking into account the nonlinearity (see e.g. the famous nested saturation solution given in [29] for the chain of integrators and generalized in [27], and [14] for another solution based on optimization).

The second aim of the studies of saturated control system deals with the robustness issue: once a saturated feedback law is designed to stabilize a control system, one generally considers the closed-loop system subject to disturbances. It turns out that the feedback law provided by minimizing time (with bounded inputs) and the nested saturated feedback law are not robust (as defined previously) (see e.g., [20]). However [21] provides a robust feedback law, which is a modification of the feedback law furnished in [14]. Other easier implementable robust feedback laws have been proposed, at least for the chain of integrators (see e.g., [3]).

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At the best of our knowledge, analysis of infinite-dimensional systems subject to saturations started with [24] and [22] and it is only very recently that appeared other works such as [13] and [19]. In [24] and [22], global asymptotic stability of the closed-loop system is tackled using nonlinear semigroup theory. Moreover [13] and [19] use similar results, but for a wider class of saturations. In [12], a nonlinear partial differential equation, namely the Korteweg-de Vries equation, is considered and some Lyapunov arguments are used to conclude on the global asymptotic stability of the closed-loop system. Finally, some nonlinear open-loop abstract control systems are studied in [11], where a LaSalle Invariance Principle is applied to prove global asymptotic stability.

A system subject to disturbances is said to be input-to-state stable (for short ISS) if its state is bounded by the norm of the disturbances. It implies that the state is bounded if the disturbances are bounded, and converges to 0 if the disturbances goes to 0. Such a notion, introduced in [26] for finite-dimensional systems, is an important tool to design robust feedback laws or even observers. This kind of properties has been studied for finite-dimensional control systems subject to saturations (see e.g., [1]). Note that the ISS notion has been extended for infinite-dimensional systems (see e.g., [18], [7] and [16]).

The aim of this paper is to study linear infinite-dimensional systems subject to saturations and disturbances, therefore to find ISS-Lyapunov functions for such systems. Two cases are considered: 1) the saturation acts in the same space as the control space; 2) the saturation acts in another space, especially a Banach space. For the first case, we provide an ISS-Lyapunov function inspired by [10]. For the second case, we provide the ISS property, without giving any explicit Lyapunov function.

The paper is organized as follows. In Section II, the problem is stated, some useful definitions are introduced and the main result is provided. In Section III, some definitions and technical lemmas are given together with the proof of the main result. Some numerical simulation are collected in Section IV for the linearized Korteweg-de Vries equation. Finally, Section V collects some concluding remarks.

II. PROBLEM STATEMENT AND MAIN RESULTS

A. Some remarks on the linear system

Let H and U be real Hilbert spaces equipped with the scalar product $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_U$, respectively. Let $A : D(A) \subset H \rightarrow H$ be a (possibly unbounded) linear operator

whose domain $D(A)$ is dense in H . Suppose that A generates a strongly continuous semigroup of linear contractions denoted by e^{tA} . Let $B : U \rightarrow H$ be a bounded operator. The operators A^* and B^* denote the adjoint operators of A and B , respectively. The aim of this paper is to study the following abstract control system

$$\begin{cases} \frac{d}{dt}z = Az + Bu, \\ z(0) = z_0. \end{cases} \quad (1)$$

Suppose that we have the following output

$$y = B^*z. \quad (2)$$

Therefore, in this case, a natural feedback law for (1) is $u = -B^*z$. One needs the following assumption.

Assumption 1. [Linear feedback law]

The feedback law $u = -B^*z$ makes the origin of the closed-loop system

$$\begin{cases} \frac{d}{dt}z = (A - BB^*)z, \\ z(0) = z_0. \end{cases} \quad (3)$$

globally exponentially stable.

It is easy to see that, under Assumption 1, $A - BB^*$ generates a strongly continuous semigroup of contractions (see [22] for a precise proof of this result). In the remaining sections of this paper, the following notation will be used.

$$\tilde{A} := A - BB^* \quad (4)$$

Let us denote the strongly continuous semigroup of contractions generated by \tilde{A} by $e^{t\tilde{A}}$.

From [5, Theorem 5.1.3, page 217], Assumption 1 holds if and only if there exist a self-adjoint and definite positive operator $P \in \mathcal{L}(H)$ and a positive value C such that

$$\langle \tilde{A}z, Pz \rangle_H + \langle Pz, \tilde{A}z \rangle_H \leq -C\|z\|_H^2, \quad \forall z \in D(\tilde{A}), \quad (5)$$

where $D(\tilde{A})$ is the domain of \tilde{A} (by (4) and the boundedness of B , it coincides with the domain of A). In other words, this inequality implies that there exists a Lyapunov function for the closed-loop system (3), given by

$$V(z) := \langle Pz, z \rangle_H. \quad (6)$$

B. Saturation functions

In this section, the case where the feedback law is bounded is studied. As explained in introduction, the feedback laws rely on the use of a saturation function defined next.

Definition 1. [Saturation functions on S] Let S be a real Banach space equipped with the norm $\|\cdot\|_S$. Assume moreover that (U, S) is a rigged Hilbert space. In other words, S is a dense subspace of U and the following inclusions hold

$$S \subseteq U \subseteq S'. \quad (7)$$

In particular, the duality pairing between S and S' is compatible with the inner product on U , in the sense that

$$\langle v, u \rangle_{S' \times S} = \langle v, u \rangle_U, \quad \forall u \in S \subset U, \quad \forall v \in U = U' \subset S'. \quad (8)$$

A function $\sigma : S \rightarrow S$ is said to be a saturation function on S admissible for U if it satisfies the following properties

1. For any $s \in S$, the following holds

$$\|\sigma(s)\|_S \leq 1; \quad (9)$$

2. For any $s, \tilde{s} \in S$, the following holds

$$\langle \sigma(s) - \sigma(\tilde{s}), s - \tilde{s} \rangle_U \geq 0; \quad (10)$$

3. For any $s, \tilde{s} \in S$, there exists a positive value k such that the following holds

$$\|\sigma(s) - \sigma(\tilde{s})\|_U \leq k\|s - \tilde{s}\|_U; \quad (11)$$

4. For any $s \in U$, the following holds

$$\|\sigma(s) - s\|_{S'} \leq \langle \sigma(s), s \rangle_U. \quad (12)$$

5. For any $s, \tilde{s} \in U$, there exists C_0 such that the following holds

$$\langle s, \sigma(s + \tilde{s}) - \sigma(s) \rangle_U \leq C_0\|\tilde{s}\|_U. \quad (13)$$

Example 1. [Examples of saturations] Here are examples of saturations borrowed from [28] or [24].

1. Suppose that $S := L^\infty(0, 1)$ and $U = L^2(0, 1)$. In this case, S' is the space of Radon measures. It contains strictly the space $L^1(0, L)$. In particular, the norm of S' is identified by the norm of $L^1(0, L)$. Hence, (U, S) is a rigged Hilbert space.

The classical saturation, which is used in practice, can be defined as follows:

$$\begin{aligned} L^\infty(0, 1) &\rightarrow L^\infty(0, 1), \\ s &\mapsto \mathbf{sat}_{L^\infty(0,1)}(s), \end{aligned} \quad (14)$$

where

$$\mathbf{sat}_{L^\infty(0,1)}(s)(x) = \mathbf{sat}_{\mathbb{R}}(s(x)) \quad (15)$$

and

$$\mathbf{sat}_{\mathbb{R}}(s) := \begin{cases} -1 & \text{if } s \leq -1, \\ s & \text{if } -1 \leq s \leq 1, \\ 1 & \text{if } s \geq 1, \end{cases} \quad (16)$$

All the items of Definition 1 are proved in [8] and [28].

2. Suppose that $S := U$. The saturation, studied in [24], [9] and [12], is defined as follows, for all $s \in U$

$$\mathbf{sat}_U(s) := \begin{cases} s & \text{if } \|s\|_U \leq 1, \\ \frac{s}{\|s\|_U} & \text{if } \|s\|_U \geq 1. \end{cases} \quad (17)$$

This is obvious that this operator satisfies Item 1. of Definition 1. In [22], this operator is proved to be m -dissipative, which implies that it satisfies Item 2. of Definition 1. The fact that this operator is globally Lipschitz is proven in [24]. Hence, this operator satisfies Item 3.

Let us prove that \mathbf{sat}_U satisfies Item 4. of Definition 1.

Case 1: $\|s\|_U \leq 1$.

In this case, one has $\mathbf{sat}_U(s) = s$ and

$$\|\mathbf{sat}_U(s) - s\|_U = 0 \leq \langle \mathbf{sat}_U(s), s \rangle_U = \|s\|_U^2. \quad (18)$$

Case 2: $\|s\|_U \geq 1$.

In this case, the following holds

$$\mathfrak{sat}_U(s) = \frac{s}{\|s\|_U} \quad (19)$$

Hence, it is immediate to see that, for all $s \in U$ satisfying $\|s\|_U \geq 1$, one has

$$\|\mathfrak{sat}_U(s) - s\|_U \leq \langle \mathfrak{sat}_U(s), s \rangle_U \leq \|s\|_U^2. \quad (20)$$

Hence, $s \in U \rightarrow \mathfrak{sat}(s) \in U$ satisfies Item 4 of Definition 1.

Let us prove that $s \in U \rightarrow \mathfrak{sat}(s) \in U$ satisfies Item 5. of Definition 1. The proof is inspired by [24].

Case 1: $\|s\|_U \leq 1$, $\|s + \tilde{s}\|_U \leq 1$.

In this case, the following holds

$$\begin{aligned} \langle s, \mathfrak{sat}_U(s + \tilde{s}) - \mathfrak{sat}_U(s) \rangle_U &= \langle s, \tilde{s} \rangle_U \\ &\leq \|\tilde{s}\|_U \end{aligned}$$

Case 2: $\|s\|_U \geq 1$, $\|s + \tilde{s}\|_U \geq 1$

$$\begin{aligned} \langle s, \mathfrak{sat}_U(s + \tilde{s}) - \mathfrak{sat}_U(s) \rangle_U &= \left\langle s, \frac{s + \tilde{s}}{\|s + \tilde{s}\|_U} - \frac{s}{\|s\|_U} \right\rangle_U \\ &\leq \|s\|_U \left\| \frac{s + \tilde{s}}{\|s + \tilde{s}\|_U} - \frac{s}{\|s\|_U} \right\|_U \end{aligned}$$

We have

$$\begin{aligned} \left\| \frac{s + \tilde{s}}{\|s + \tilde{s}\|_U} - \frac{s}{\|s\|_U} \right\|_U &= \left\| \frac{(s + \tilde{s})\|s\|_U - s\|s + \tilde{s}\|_U}{\|s + \tilde{s}\|_U \|s\|_U} \right\|_U \\ &\leq \frac{1}{\|s\|_U \|s + \tilde{s}\|_U} 2\|s + \tilde{s}\|_U \|s + \tilde{s} - s\|_U \\ &\leq \frac{2}{\|s\|_U} \|\tilde{s}\|_U \end{aligned}$$

Therefore, one has

$$\langle s, \mathfrak{sat}_U(s + \tilde{s}) - \mathfrak{sat}_U(s) \rangle_U \leq 2\|\tilde{s}\|_U \quad (21)$$

Case 3: $\|s\| \leq 1$, $\|s + \tilde{s}\|_U \geq 1$

Using the fact that $s \mapsto \mathfrak{sat}_U(s)$ is globally Lipschitz with constant 3 yields

$$\begin{aligned} \langle s, \mathfrak{sat}_U(s + \tilde{s}) - \mathfrak{sat}_U(s) \rangle_U &\leq \|s\|_U \|\mathfrak{sat}_U(s + \tilde{s}) - \mathfrak{sat}_U(s)\|_U \\ &\leq 3\|\tilde{s}\|_U. \end{aligned}$$

Case 4: $\|s\| \geq 1$, $\|s + \tilde{s}\|_U \leq 1$ In this case, one has

$$\begin{aligned} \langle s, \mathfrak{sat}_U(s + \tilde{s}) - \mathfrak{sat}_U(s) \rangle_U &= \left\langle s, s + \tilde{s} - \frac{s}{\|s\|_U} \right\rangle_U \\ &\leq \|s\|_U \left\| \frac{(s + \tilde{s})\|s\| - s}{\|s\|_U} \right\|_U \\ &\leq \|s + \tilde{s}\| (3\|s + \tilde{s} - s\|_U) \\ &\leq 3\|\tilde{s}\|_U \end{aligned}$$

Therefore \mathfrak{sat}_U satisfies all items of Definition 1 for an admissible saturation map.

C. Discussion on the well-posedness

1) *Case without disturbance:* We aim at studying the following closed-loop system

$$\begin{cases} \frac{d}{dt}z = Az - B\sigma(B^*z) := A_\sigma z, \\ z(0) = z_0, \end{cases} \quad (22)$$

where the operator A_σ is defined as follows

$$\begin{aligned} A_\sigma : D(A_\sigma) \subset H &\rightarrow H \\ z &\mapsto A_\sigma z, \end{aligned} \quad (23)$$

with $D(A_\sigma) = D(A)$.

From [23, Lemma 2.1., Part IV, Page 165.], we have the following proposition

Proposition 1. [Well-posedness] *The operator A_σ generates a strongly continuous semigroup of contractions, which we denote by $(W_\sigma(t))_{t \geq 0}$. Hence, there exists a unique strong solution to (22) and, for all $z_0 \in D(A_\sigma)$, the following functions*

$$t \mapsto \|W_\sigma(t)z_0\|_H, \quad t \mapsto \|A_\sigma W_\sigma(t)z_0\|_H \quad (24)$$

are nonincreasing.

As far as the well-posedness of the closed-loop system (22) is concerned, the interested reader can read [22], [24] or [11]. The case under consideration in Proposition 1, the proof is easier than in the previous work, since the nonlinear operators σ considered in Definition 1 are globally Lipschitz and monotone. The proof of Proposition 1 is omitted to focus, in this conference paper, on the main result, namely the stability and ISS properties of a nonlinear infinite-dimensional systems, and its proofs.

2) *Case with disturbance:* In this paper, we will also consider the case with a perturbation. The system under consideration is the following

$$\begin{cases} \frac{d}{dt}z = Az - B\sigma(B^*z + d), \\ z(0) = z_0, \end{cases} \quad (25)$$

where $d : \mathbb{R}_{\geq 0} \rightarrow U$ is the perturbation. Note that d belongs to $\mathfrak{A} := L^2((0, \infty); U)$.

For this system, we study only weak solution, that can be written as follows, for every $z_0 \in H$.

$$z(t) = e^{tA}z_0 - \int_0^t e^{(t-s)A} B\sigma(B^*z(s) + d(s))ds. \quad (26)$$

Let us introduce two notions related to the ISS property.

Definition 2. [ISS property]

1. *The origin of (25) is said to be ISS with respect to d if there exist a class \mathcal{KL} function α and a class \mathcal{K}_∞ function ρ such that, for any weak solution to (25)*

$$\|z\|_H \leq \alpha(\|z_0\|_H, t) + \rho(\|d\|_{\mathfrak{A}}), \quad \forall t \geq 0.$$

2. *A positive definite function $V : H \rightarrow \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function with respect to d if there exist two*

class \mathcal{K}_∞ functions α and ρ such that, for any weak solution to (25)

$$\frac{d}{dt}V(z) \leq -\alpha(\|z\|) + \rho(\|d\|_U).$$

Remark 1. • Note that the Lyapunov function V might be non-coercive. In this case, in general, Item 2. of Definition 2 does not imply the ISS property (see e.g., [16]). Another property is needed, which is the following: for any positive values C and τ , it holds that

$$\sup\{\|z\| \mid \|z_0\|_H \leq C, \|d\|_H \leq C, t \in [0, \tau]\} < \infty. \quad (27)$$

In the special case of system (25), this property holds. Checking it reduces to performing the derivative of the function $\|z\|_Z^2$ along the solutions to (25) and to using Item 5. of Definition 1. Indeed, one obtains

$$\frac{d}{dt}\|z\|_H^2 \leq C_0\|d\|_U, \quad (28)$$

which implies

$$\|z\|_H^2 \leq \|z_0\|_H^2 + C_0\|d\|_U. \quad (29)$$

This latter inequality implies the property defined by (27).

- It is not sure yet that Item 1. implies the existence of an ISS Lyapunov function (see e.g., [15] and [16]).

D. Input-to-state stability result

We are now in position to state our main result. Here is its statement.

Theorem 1. [ISS result]

Suppose that Assumption 1 holds and let $P \in \mathcal{L}(H)$ be the self-adjoint and positive operator $P \in \mathcal{L}(H)$ satisfying (5). Let σ be a saturation function satisfying all the items of Definition 1. Then, the following holds.

1. If $S = U$, there exists an ISS-Lyapunov function for (25).
2. If $S \neq U$, assume that the following holds, for any $s \in D(A)$,

$$\|B^*s\|_S \leq c_S\|s\|_{D(A)}. \quad (30)$$

Then, the origin of (25) is ISS with respect to d .

III. PROOF OF THE MAIN RESULT

A. Definitions and technical lemmas

Before proving Theorem 1, let us give some useful definitions and results to prove Theorem 1, especially the case where $S \neq U$.

The following result links the disturbed system (25) to the undisturbed one (22).

Lemma 1. If 0 is globally asymptotically stable for the following abstract control system, for any $z_0 \in H$

$$\begin{cases} \frac{d}{dt}z = Az - B\sigma(B^*z), \\ z(0) = z_0, \end{cases} \quad (31)$$

then 0 is ISS with respect to d for the following abstract control system, for any $z_0 \in H$

$$\begin{cases} \frac{d}{dt}z^d = Az^d - B\sigma(B^*z^d + d), \\ z^d(0) = z_0. \end{cases} \quad (32)$$

Let us prove this lemma.

Proof of Lemma 1: Let us introduce $\tilde{z} := z^d - z$. It satisfies the following abstract system

$$\begin{cases} \frac{d}{dt}\tilde{z} = A\tilde{z} + B\sigma(B^*z) - B\sigma(B^*z^d + d), \\ z(0) = 0. \end{cases} \quad (33)$$

Moreover, we compute

$$\begin{aligned} \frac{d}{dt}\|\tilde{z}\|_H^2 &= \langle A\tilde{z}, \tilde{z} \rangle_H + \langle \tilde{z}, A\tilde{z} \rangle_H \\ &\quad + 2\langle B(\sigma(B^*z) - \sigma(B^*z^d + d)), \tilde{z} \rangle_H \\ &\leq \|\sigma(B^*z) - \sigma(B^*z^d + d)\|_U \|B^*\tilde{z}\|_U \\ &\leq k\|B^*\tilde{z} + d\|_U \|B^*\tilde{z}\|_U \\ &\leq k(\|B^*\tilde{z}\|_U + \|d\|_U) \|B^*\tilde{z}\|_U \\ &\leq \frac{3}{2}k\|B^*\|_{\mathcal{L}(H,U)}^2 \|\tilde{z}\|_U^2 + \frac{1}{2}k\|d\|_U^2, \end{aligned}$$

where (33) has been used for the first inequality, Item 3. of Definition 1 has been used to get the second one, the triangle inequality has been used in the third one, and Cauchy-Schwarz inequality has been used for the last one. Applying Grönwall Lemma, one obtains

$$\|\tilde{z}(t)\|_H^2 \leq e^{\frac{3}{2}k\|B^*\|_{\mathcal{L}(H,U)}^2 t} \|\tilde{z}_0\|_H^2 + \frac{1}{2}k \int_0^t \|d(\tau)\|_U^2 d\tau$$

where \tilde{z}_0 is the initial condition of \tilde{z} . Since the initial condition of \tilde{z} is null, it follows

$$\|\tilde{z}(t)\|_H \leq \sqrt{\frac{1}{2}k \int_0^t \|d(\tau)\|_U^2 d\tau}$$

Using the fact that $\tilde{z} = z^d - z$ and a triangle inequality yields

$$\|z^d\|_H \leq \|z\|_H + \sqrt{\frac{1}{2}k \int_0^t \|d(\tau)\|_U^2 d\tau}$$

Now, note that if the origin of (31) is globally asymptotically stable, then there exists a class \mathcal{KL} α such that

$$\|z\|_H \leq \alpha(\|z_0\|_H, t), \quad \forall t \geq 0. \quad (34)$$

Therefore, it implies

$$\|z^d\|_H \leq \alpha(\|z_0\|_H, t) + \rho(\|d\|_U), \quad (35)$$

where ρ is given by: $\rho(s) := \sqrt{\frac{1}{2}ks}$. This concludes the proof of Lemma 1. \square

Thanks to this result, proving ISS for (25) reduces to proving global asymptotic stability of the origin of (22) for every weak solution to (22). However, when $S \neq U$, the properties given in Definition 1 do not hold when considering weak solutions. Therefore, the following result is needed to link the attractivity for strong and weak solutions. Note that

its proof, which relies on a density argument, can be found in [9] and [11].

Lemma 2. *Let $(W_\sigma(t))_{t \geq 0}$ be a strongly continuous semigroup of contractions on H , a Hilbert space. Let $D(A)$ be dense in H . If for all $z_0 \in D(A)$, the following holds*

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)z_0\|_H = 0, \quad (36)$$

then, for all $z_0 \in H$,

$$\lim_{t \rightarrow +\infty} \|W_\sigma(t)z_0\|_H = 0. \quad (37)$$

Thanks to this latter lemma, proving the global asymptotic stability of the origin of (22) for every weak solution reduces to proving it for every strong solution to (22). We need another definition of stability before proving the global asymptotic stability.

Definition 3. *The origin of (22) is said to be semi-globally exponentially stable if, for any positive value r and any initial condition satisfying $\|z_0\|_{D(A_\sigma)} \leq r$, there exist two positive values $\mu := \mu(r)$ and $K := K(r)$ such that, for every strong solution to (22)*

$$\|W_\sigma(t)z_0\|_H \leq Ke^{-\mu t} \|z_0\|_H, \quad \forall t \geq 0, \quad (38)$$

where $(W_\sigma(t))_{t \geq 0}$ is the strongly continuous semigroup of contractions generated by the operator A_σ .

This definition is inspired by [12]. One has also the following result.

Lemma 3. *If the origin of (22) is semi-globally exponentially stable, then it is globally asymptotically stable.*

Proof of Lemma 3: Consider a strong solution \tilde{z} to (22) whose initial condition is such that

$$\|\tilde{z}_0\|_{D(A)} \leq 1. \quad (39)$$

Since the origin of (22) is semi-globally exponentially stable, there exist two positive values μ_1 and K_1 , such that

$$\|W_\sigma(t)\tilde{z}_0\|_H \leq K_1 e^{-\mu_1 t} \|\tilde{z}_0\|_H, \quad \forall t \geq 0. \quad (40)$$

Now, consider a strong solution z to (22) whose initial condition is such that,

$$\|z_0\|_{D(A)} \leq r, \quad (41)$$

where r is a positive value. Hence, since the origin of (22) is semi-globally exponentially stable, there exist two positive values μ_r and K_r such that

$$\|W_\sigma(t)z_0\|_H \leq K_r e^{-\mu_r t} \|z_0\|_H, \quad \forall t \geq 0. \quad (42)$$

Consequently, setting $T_r = \mu_r^{-1} \ln(rK_r)$, it holds from (42)

$$\|z_0\|_H \leq r \Rightarrow \|W_\sigma(T_r)z_0\|_H \leq e^{-\ln(rK_r)} rK_r = 1 \quad (43)$$

Therefore, using (40) and the fact that $t \mapsto W_\sigma(t)$ is a semigroup of contractions, it follows

$$\begin{aligned} \|W_\sigma(t)z_0\|_H &\leq K_1 e^{-\mu_1(t-T_r)} \|W_\sigma(T_r)z_0\|_H, \quad \forall t \geq T_r \\ &\leq K_1 e^{\mu_1 T_r} e^{-\mu_1 t} \|z_0\|_H. \end{aligned}$$

This concludes the proof of Lemma 3. \square

B. Proof of Theorem 1

Proof of Theorem 1: We split the proof of Theorem 1 into two cases. Firstly, we prove Item 1. of Theorem 1 and then Item 2. Indeed, the Lyapunov functions considered in both cases are different.

Case 1: $S = U$.

Let us consider the following candidate Lyapunov function

$$\begin{aligned} V_1(z) &:= \langle Pz, z \rangle_H + \frac{2M}{3} \|z\|_H^3 \\ &= V(z) + \frac{2M}{3} \|z\|_H^3, \end{aligned} \quad (44)$$

where $P \in \mathcal{L}(H)$ is defined in (5) and M is a sufficiently large positive value that will be chosen later. This function, inspired by [10], is clearly positive definite and tends to infinity if the H -norm of z does.

Firstly applying Cauchy-Schwarz inequality, one has, along the strong solutions to (25),

$$\begin{aligned} \frac{d}{dt} V(z) &= \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H \\ &\quad + \langle PB(B^*z - \sigma(B^*z)), z \rangle_H \\ &\quad + \langle Pz, B(B^*z - \sigma(B^*z)) \rangle_H \\ &\quad + \langle PB(\sigma(B^*z) - \sigma(B^*z + d)), z \rangle_H \\ &\quad + \langle z, PB(\sigma(B^*z) - \sigma(B^*z + d)) \rangle_H \\ &\leq -C \|z\|_H^2 + 2\|B^*z\|_U \|P\|_{\mathcal{L}(H)} \|B^*z - \sigma(B^*z)\|_U \\ &\quad + 2\langle \sigma(B^*z) - \sigma(B^*z + d), B^*Pz \rangle_U, \\ &\leq -C \|z\|_H^2 + 2\|B^*z\|_U \|P\|_{\mathcal{L}(H)} \|B^*z - \sigma(B^*z)\|_U \\ &\quad + 2k\|d\|_U \|B^*\|_{\mathcal{L}(H,U)} \|P\|_{\mathcal{L}(H)} \|z\|_H, \end{aligned}$$

where (5) and Item 3. of Definition 1 have been used to get this inequality. Using Item 4. of Definition 1, Cauchy-Schwarz inequality and the fact that B^* is bounded in U yields

$$\begin{aligned} \frac{d}{dt} V(z) &\leq - \left(C - \frac{\|B^*\|_{\mathcal{L}(H,U)}^2 \|P\|_{\mathcal{L}(H)}^2}{\varepsilon_1} \right) \|z\|_H^2 \\ &\quad + 2\|B^*z\|_U \|P\|_{\mathcal{L}(H)} \|B^*z - \sigma(B^*z)\|_U \\ &\quad + k^2 \varepsilon_1 \|d\|_U^2, \\ &\leq - \left(C - \frac{\|B^*\|_{\mathcal{L}(H,U)}^2 \|P\|_{\mathcal{L}(H)}^2}{\varepsilon_1} \right) \|z\|_H^2 \\ &\quad + 2\|B^*\|_{\mathcal{L}(H,U)} \|P\|_{\mathcal{L}(H)} \|z\|_H (\langle \sigma(B^*z), B^*z \rangle_U \\ &\quad + \langle B^*z, \sigma(B^*z) \rangle_U) \\ &\quad + k^2 \varepsilon_1 \|d\|_U^2 \\ &\leq - \left(C - \frac{\|B^*\|_{\mathcal{L}(H,U)}^2 \|P\|_{\mathcal{L}(H)}^2}{\varepsilon_1} \right) \|z\|_H^2 \\ &\quad + 2\|B^*\|_{\mathcal{L}(H,U)} \|P\|_{\mathcal{L}(H)} \|z\|_H (\langle \sigma(B^*z), B^*z \rangle_U \\ &\quad + \langle B^*z, \sigma(B^*z) \rangle_U) + k^2 \varepsilon_1 \|d\|_U^2, \end{aligned}$$

where ε_1 is a positive value that will be selected later. Secondly, using the dissipativity of the operator A_σ , Item

5. of Definition 1 and Young inequality, one has

$$\begin{aligned}
\frac{2M}{3} \frac{d}{dt} \|z\|_H^3 &= M \|z\|_H (\langle Az, z \rangle_H + \langle z, Az \rangle_H) \\
&\quad - 2M \|z\|_H \langle B\sigma(B^*z + d), z \rangle_H \\
&\leq -2M \|z\|_H (\langle \sigma(B^*z), B^*z \rangle_U \\
&\quad + \langle \sigma(B^*z) - \sigma(B^*z + d), B^*z \rangle_U) \\
&\leq -2M \|z\|_H (\langle \sigma(B^*z), B^*z \rangle_U) \\
&\quad + 2MC_0 \|z\|_H \|d\|_U \\
&\leq -2M \|z\|_H (\langle \sigma(B^*z), B^*z \rangle_U) \\
&\quad + \frac{2MC_0}{\varepsilon_2} \|z\|_H^2 + 2MC_0 \varepsilon_2 \|d\|_U^2,
\end{aligned}$$

where ε_2 is a positive value that has to be selected. Hence, if one chooses M , ε_1 and ε_2 as follows

$$\begin{cases} M \geq 2 \|B^*\|_{\mathcal{L}(H,U)} \|P\|_{\mathcal{L}(H)}, \\ \frac{2MC_0}{\varepsilon_2} + \frac{\|B^*\|_{\mathcal{L}(H,U)}^2 \|P\|_{\mathcal{L}(H)}^2}{\varepsilon_1} \leq C, \end{cases}$$

one obtains

$$\begin{aligned}
\frac{d}{dt} V_1(z) &\leq - \left(C - \frac{2M}{\varepsilon_2} - \frac{\|B^*\|_{\mathcal{L}(H,U)}^2 \|P\|_{\mathcal{L}(H)}^2}{\varepsilon_1} \right) \|z\|_H^2 \\
&\quad + (C_0 2M \varepsilon_2 + k^2 \varepsilon_1) \|d\|_U^2.
\end{aligned}$$

With an appropriate choice of ε_1 and ε_2 , it concludes the proof of Item 1. of Theorem 1.

Case 2: $S \neq U$.

As it has been noticed before, proving ISS reduces to proving the global asymptotic stability of the origin thanks to Lemma 1. Therefore, in the case where $S \neq U$, we will prove that the origin of (22) is semi-globally exponentially stable. Using Lemma 3, it proves that the origin of (22) is globally asymptotically stable.

Pick a positive value r . Consider any strong solution to (22) starting from z_0 satisfying

$$\|z_0\|_{D(A)} \leq r. \quad (45)$$

Now, focus on the following Lyapunov function

$$\begin{aligned}
V_2(z) &:= \langle Pz, z \rangle_H + \tilde{M}r \|z\|_H^2 \\
&= V(z) + \tilde{M}r \|z\|_H^2
\end{aligned} \quad (46)$$

where \tilde{M} is a sufficiently large positive value that will be selected later.

Firstly, using Assumption 1, one has

$$\begin{aligned}
\frac{d}{dt} V(z) &= \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H \\
&\quad + \langle PB(B^*z - \sigma(B^*z)), z \rangle_H \\
&\quad + \langle Pz, B(B^*z - \sigma(B^*z)) \rangle_H \\
&\leq -C \|z\|_H^2 + \langle B^*z - \sigma(B^*z), B^*Pz \rangle_U \\
&\quad + \langle B^*Pz, B^*z - \sigma(B^*z) \rangle_U.
\end{aligned}$$

Since (U, S) is a rigged Hilbert space, one has the following equality

$$\begin{aligned}
\langle B^*z - \sigma(B^*z), B^*Pz \rangle_U + \langle B^*Pz, B^*z - \sigma(B^*z) \rangle_U = \\
2 \langle B^*z - \sigma(B^*z), B^*Pz \rangle_{S' \times S}.
\end{aligned}$$

Hence, applying Cauchy-Schwarz's inequality, one obtains

$$\frac{d}{dt} V(z) \leq -C \|z\|_H^2 + 2 \|B^*Pz\|_S \|B^*z - \sigma(B^*z)\|_{S'}$$

Using Item 4. of Definition 1 yields

$$\begin{aligned}
\frac{d}{dt} V(z) &\leq -C \|z\|_H^2 \\
&\quad + 4c_S \|Pz\|_{D(A)} (\langle B^*z, \sigma(B^*z) \rangle_U), \quad (47)
\end{aligned}$$

where, in the second line, it has been used the assumption given in (30).

Now, using (45) together with the monotonicity of the function in (24), along the strong solutions to (22), we have

$$\|W_\sigma(t)z_0\|_{D(A)} \leq \|z_0\|_{D(A)}. \quad (48)$$

Therefore, noticing that $\|Pz\|_{D(A)} := \|Pz\|_H + \|PAz\|_H \leq \|P\|_{\mathcal{L}(H)} \|z\|_{D(A)}$, and using (47) and (30), one obtains

$$\begin{aligned}
\frac{d}{dt} V(z) &\leq -C \|z\|_H^2 \\
&\quad + 4c_S r \|P\|_{\mathcal{L}(H)} (\langle z, B\sigma(B^*z) \rangle_H).
\end{aligned}$$

Moreover, one has, using the dissipativity of the operator A

$$\begin{aligned}
\frac{d}{dt} \tilde{M}r \|z\|_H^2 &= \tilde{M}r \langle Az, z \rangle_H + \langle z, Az \rangle_H \\
&\quad - \tilde{M}r (\langle z, B\sigma(B^*z) \rangle_H + \langle B\sigma(B^*z), z \rangle_H) \\
&\leq -4\tilde{M}r (\langle z, B\sigma(B^*z) \rangle_H).
\end{aligned}$$

Hence, if one selects \tilde{M} such that

$$\tilde{M} > 2c_S \|P\|_{\mathcal{L}(H)},$$

it follows

$$\frac{d}{dt} V_2(z) \leq -C \|z\|_H^2. \quad (49)$$

Note that, from (46), we have, for all $z \in H$,

$$\tilde{M}r \|z\|_H^2 \leq V_2(z) \leq (\|P\|_{\mathcal{L}(H)} + \tilde{M}r) \|z\|_H^2 \quad (50)$$

It yields

$$\frac{d}{dt} V_2(z) \leq - \frac{C}{\|P\|_{\mathcal{L}(H)} + \tilde{M}r} V_2(z) \quad (51)$$

Applying Grönwall's inequality, one obtains

$$V_2(W_\sigma(t)z_0) \leq e^{-\mu t} V_2(z_0), \quad \forall t \geq 0, \quad (52)$$

where $\mu := \frac{C}{\|P\|_{\mathcal{L}(H)} + \tilde{M}r}$.

Therefore, using (50) yields

$$\|W_\sigma(t)z_0\|_H^2 \leq \frac{\|P\|_{\mathcal{L}(H)} + \tilde{M}r}{\tilde{M}r} e^{-\mu t} \|z_0\|_H^2. \quad (53)$$

Hence, the origin of (22) is semi-globally exponentially stable for any strong solution to (22). It yields that the origin is globally asymptotically stable from Lemma 3. Moreover, using Lemma 2, the origin of (22) is globally asymptotically stable for any weak solution to (22). Therefore, using Lemma 1, this concludes the proof of Item 2. of Theorem 1. \square

IV. EXAMPLE: A KORTEWEG-DE VRIES EQUATION

A. Applying Theorem 1

The Korteweg-de Vries equation (for short KdV) describes long waves in water of relatively shallow depth. It has been deeply studied in recent decades (see e.g., [2] for a nice introduction to the KdV equation in the context of the control).

The linearized version of the controlled KdV equation can be written as follows

$$\begin{cases} z_t + z_x + z_{xxx} = u, & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z(0, x) = z_0(x), & x \in [0, L]. \end{cases} \quad (54)$$

Given

$$H = U = L^2(0, L), \\ D(A) := \{z \in H^3(0, L) \mid z(0) = z(L) = z'(L) = 0\},$$

one can describe (54) as an abstract control system with the following operators

$$A : z \in D(A) \subset H \mapsto -z' - z''' \in H, \quad B = I_H, \quad (55)$$

where I_H denotes the identity operator for H . A simple computation shows that

$$\langle Az, z \rangle_H \leq 0, \quad \langle A^*z, z \rangle_H \leq 0.$$

These inequalities together with the fact that A is a closed operator allow us to apply the Lümer-Phillips theorem. Therefore, A generates a strongly continuous semigroup of contractions.

If one sets $u = -B^*z = -z$, then Assumption 1 holds. Indeed, the associated Lyapunov function is

$$V(z) = \frac{1}{2} \|z\|_H^2. \quad (56)$$

Its derivative along the solutions to (54) satisfies

$$\frac{d}{dt} V(z) \leq -V(z). \quad (57)$$

Therefore, the operator P given by (5) reduces to the identity.

Now, focus on the case where the control is saturated and where $S \neq U$. Let $S = L^\infty(0, L)$ and σ be given by (14). In [13], the following lemma has been proved

Lemma 4. *[[13], Lemma 4.] There exists a positive value Δ such that, for all $z \in D(A)$,*

$$\|z\|_{H_0^1(0, L)} \leq \Delta \|z\|_{D(A)}. \quad (58)$$

Using the above mentioned result together with the fact that the space $H_0^1(0, L)$ is continuously embedded in $L^\infty(0, L)$, one obtains that

$$\|z\|_S \leq \Delta \|z\|_{D(A)}. \quad (59)$$

Since $B^* = I_H$, one has

$$\|B^*z\|_S \leq \Delta \|z\|_{D(A)}. \quad (60)$$

Therefore, (30) holds for the linear Korteweg-de Vries equation (54).

Finally, since all the properties needed to apply Theorem 1 hold, this proves that the origin of

$$\begin{cases} z_t + z_x + z_{xxx} = -\text{sat}_{L^\infty(0, L)}(z + d), & (t, x) \in \mathbb{R}_{\geq 0} \times [0, L], \\ z(t, 0) = z(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z_x(t, L) = 0, & t \in \mathbb{R}_{\geq 0} \\ z(0, x) = z_0(x), & x \in [0, L], \end{cases} \quad (61)$$

is ISS with respect to d .

B. Numerical simulations

Let us discretize the PDE (61) by means of finite difference method (see e.g. [17] for an introduction on the numerical scheme of a generalized Korteweg-de Vries equation). The time and the space steps are chosen such that the stability condition of the numerical scheme is satisfied.

We choose $L = 2\pi$, $T = 9$, $z_0(x) = (1 - \cos(x))$ for all $x \in [0, 2\pi]$ and $d(t) = 0.05 \cos(t)$. Let us numerically compute the corresponding solution to (61).

Figure 1 (top) gives the time-evolution of this solution to (61), where the convergence property could be checked. Since the disturbance is bounded, one can check that the solution is also, which is one of the property of ISS. Figure IV-B illustrates the Lyapunov function $\|z\|_{L^2(0, L)}^2$ with respect to the time and the solution without perturbation and without saturation. One can see that the L^2 -norm of the solution to (61) is bounded.

V. CONCLUSION

In this paper, the analysis of a stabilizing feedback law modified via a saturation function has been tackled using Lyapunov theory for infinite-dimensional systems. The saturation under consideration might render the feedback-law bounded in another space than the one where the origin is stabilized. Assuming a stabilizability property and some regularity for the Lyapunov and the control operators, the system has been proved to be ISS.

For future research lines, it would be interesting to study the case of unbounded control operators as considered for instance in [4], [30] or [19]. The case of nonlinear open-loop systems, as it is considered in [11], could be also challenging.

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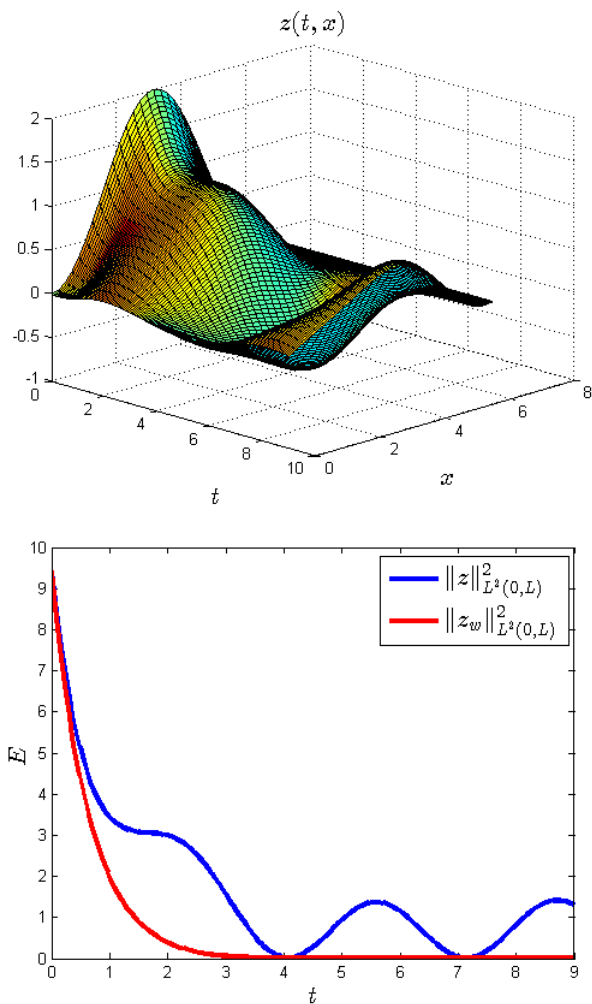


Fig. 1. Top: time-evolution of the solution to (61). Down: L^2 -norm of the solution to (61) (in blue) and of the solution to (61) without saturation and without perturbation (in red).

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