Full-Pose Tracking Control for Aerial Robotic Systems
With Laterally Bounded Input Force
Antonio Franchi, Ruggero Carli, Davide Bicego, Markus Ryll

To cite this version:
Antonio Franchi, Ruggero Carli, Davide Bicego, Markus Ryll. Full-Pose Tracking Control for Aerial
Robotic Systems With Laterally Bounded Input Force. IEEE Transactions on Robotics, IEEE, 2018,
34 (2), pp.534-541. <10.1109/TRO.2017.2786734>. <hal-01710770>
Full-Pose Tracking Control for Aerial Robotic Systems with Laterally-Bounded Input Force

Antonio Franchi, Senior Member, IEEE, Ruggero Carli, Member, IEEE, Davide Bicego, and Markus Ryll, Member, IEEE

Abstract—A class of abstract aerial robotic systems is introduced, the Laterally Bound Force (LBF) vehicles, in which most of the control authority is expressed along a principal thrust direction, while along the lateral directions a (smaller and possibly null) force may be exploited to achieve full-pose tracking. This class approximates platforms endowed with non-collinear rotors that can modify the orientation of the total thrust in body frame. The proposed SE(3)-based control strategy achieves, if made possible by the force constraints, the independent tracking of position-plus-orientation trajectories. The method, which is proven using a Lyapunov technique, deals seamlessly with both under- and fully-actuated platforms, and guarantees at least the position tracking in the case of an unfeasible full-pose reference trajectory. Several experimental tests are presented, that clearly show the approach practicability and the sharp improvement with respect to state-of-the-art.

I. INTRODUCTION

In the last years we have seen advancements in control, perception and actuation allowing unmanned aerial vehicles (UAVs) to perform very agile maneuvers [1], as well as complex missions solely or in swarms and heterogeneous groups. Originally performed missions have been contactless as, e.g., environmental monitoring and search and rescue. Nowadays UAVs are as well utilized as aerial robots to perform direct physical interaction – UAVs grasp, transport, and manipulate our environment, as shown by several research projects on aerial manipulation 1, and can also physically interact through cables [2], [3] in order to perform unprecedented maneuvers [4].

Standard multi-rotors (quadrotors, hexarotors, etc.) have collinear propellers generating forces that are all aligned to one direction in body frame, which makes them under-actuated systems: the total control force can be produced only along that direction. In order to follow a generic position trajectory the total force direction in world frame is changed by rotating the whole vehicle. Maneuvers in which rotation and translation are completely independent are precluded to such platforms, which constitutes a serious problem in the case that, e.g., the platform has to move through a hostile and cluttered ambient or resist a wind gust while keeping a desired attitude. Such an underactuation even deteriorates the potentiality to interact with the environment by rapidly exerting forces in an arbitrarily-chosen direction of the space while keeping a pre-specified orientation.

The major solution to overcome the aforementioned issues has been to mount rotors in a tilted way such that the thrusts of the propellers are not collinear anymore. In this way, the direction of the total force can be changed by selecting the intensity of the force produced by each propeller, without the need of reorienting the whole vehicle. If the propellers are at least six, and the tilting directions do not generate a singular configuration, then direction and intensity of both the instantaneous total control moment and total control force are controllable at will. This idea has been implemented in several real platforms and is becoming more and more popular in the aerial vehicles and robotics communities. For example, results achieved in [5] show an improvement in resisting an opposing wrench, while the works done in [6]–[8] show that such idea allows to decouple the tracking of a desired position and orientation.

These new kind of multi-directional thrust platforms (sometimes referred to as fully-actuated) call for new methods to control them efficiently and to reliably cope with the added complexity of the platforms and of the larger set of tasks in which they can be employed, when compared to standard collinear multi-rotors. To fill this gap, in this paper we propose a novel method for controlling such platforms while taking into account the most limiting input bounds they have to cope with, i.e., lateral input force. The proposed controller ensures, in nominal conditions, the tracking of a full-6D pose reference trajectory (position plus orientation). If the reference orientation and the force needed to track the position trajectory do not comply with the platform constraints, the proposed strategy gives priority to the tracking of the positional part while also tracking the feasible orientation that is the closest to the reference one. This choice is supported by, e.g., the fact that in typical applications a wrong position tracking is more likely to lead to an obstacle crash than a non-perfect orientation tracking.

In order to attain generality, the proposed method is designed to work with a large variety of different platforms which include not only the fully-actuated but also the under-actuated (collinear propeller) case. In this way we both maximize the breadth of the impact of the proposed methodology to new fully-actuated platforms and we also maintain backward-compatibility with standard platforms. This last feature makes

the proposed method also suitable to control vectored-thrust vehicles that can transit from an under-actuated to a fully-actuated configuration while flying – as, e.g., the one presented in [9]. The proposed method is universal and does not need any switching between two different controllers for each configuration.

Additionally, we envision that the proposed method will find large application in the new emerging topic of aerial robotic physical interaction. A striking example of this application has been shown in [10], where an admittance control framework has been built around the controller developed in this paper, and real experiments in contact with the environment are shown.

Motivations and State of the Art: To optimize the use of available energy, common multi-rotor platforms are designed with all collinear rotors, at the expense of under-actuation. For these platforms several controllers have been proposed like, e.g., the ones based on the dynamic feedback linearization [11], cascaded/backstepping ones [12], [13], and geometric ones on \( SE(3) \) [14], [15] (see [16] for a review).

Recently, new concepts have been developed where the use of non-collinear propellers [5], [6], [17] allows the orientation of the total thrust to deviate from its principal direction. However, in order to minimize the waste of energy caused by the appearance of internal forces, the maximum component of the total thrust along the lateral direction is typically kept (by design) much lower than the maximum allowed component along the vertical one. We call these kind of platforms aerial vehicles with laterally-bounded force (LBF): they are characterized by a principal direction of thrust about which most of the thrust can be exerted. A certain amount of thrust (typically smaller) can be exerted along any non-principal (lateral) directions. This model includes: i) the standard underactuated multi-rotor vehicle where thrust is possible only along the principal direction, and ii) the isotropically fully-actuated platforms where a large amount of total thrust in the lateral directions is applicable [7], [8].

An underactuated LBF platform is not able to track a generic full-pose trajectory, i.e., with independent position and orientation in \( SE(3) \). The rotation about any axis that is orthogonal to the principal fixed total thrust direction must follow the evolution over time of the position trajectory, according to the well-known differential flatness property [1], [11]. Therefore an underactuated multi-rotor aerial platform can only track a 4D-pose trajectory (i.e., position plus the rotation about the principal direction). On the contrary, a ‘fully-actuated’ LBF platform can exert some force in the lateral direction thus allowing the tracking of a generic full-pose (6D) trajectory. However, due to the bounded thrust along the lateral directions, it is not possible to track any full-pose trajectory. The larger the bounds the higher the ability of the platform to track any trajectory, the lower the bounds the more the platform resembles an underactuated multi-rotor and thus it becomes almost unable to track a full-pose trajectory but only a 4D-pose one.

The easiest approach to control fully-actuated platforms is the inverse dynamics approach. First, a control wrench is computed in order to track the desired trajectory by cancelling the nonlinear dynamical effects and trying to zero the position and orientation errors. Then the thrust inputs for each propeller are computed from the control wrench by simply inverting the control allocation matrix. This method has been first proposed in [6] and then used also in [7] (with pseudo-inversion, in place of inversion, to allocate the input redundancy) and in [8]. The limitation of this control approach is to neglect input saturation, which may easily lead to an unstable behavior if, e.g., the full-pose trajectory to be followed is not input-feasible. A preliminary attempt to solve these issue in a special case has been done in [18], where a weighted pseudo-inversion is used to allocate the 6 inputs of a coplanar hexarotor in order to obtain a lower dimensional task (4D) output (the total thrust and the 3 moment components). Another control approach is presented in [17], which is however specific to that octorotor platform, it does not consider input bounds either, and is based on a particular Euler angle representation. Finally, [19] presents an approach that also uses Euler and is based on the reference governor. The method is tested only in simulation and for constant position and attitude references.

In this paper we present a geometric tracking controller for time-varying references that is instead very general and applicable to any LBF vehicle, thus also taking into account the bounds on the lateral control force. The method is not prone to local orientation representation singularities since it is natively designed in \( SE(3) \).

The remaining part of the paper is structured as follows. Section II presents the generic model. The proposed control method is introduced in Sec. III. Section IV shows the full computation of the generic controller in a meaningful case, while results of several experiments are shown in Sec. V. Finally we conclude the paper and give an outline of further possible extensions in Sec. VI.

II. LATERALLY-BOUNDED FORCE AERIAL VEHICLES

An LBF platform is a rigid body to which gravity and control generalized forces are applied. For the sake of compactness, all the main symbols used in the paper are summarized in Table I and Fig. 1.

The vehicle orientation kinematics is described by \( \dot{\mathbf{R}} = \mathbf{R} \omega \) where \( \omega \in so(3) \) is the skew symmetric matrix associated to \( \omega \). The control inputs \( \mathbf{u}_1 = [u_1 u_2 u_3]^T \in \mathbb{R}^3 \) and \( \mathbf{u}_2 = [u_4 u_5 u_6]^T \in \mathbb{R}^3 \) are the input force and moment applied
Inverting (2), the nominal inputs to track \( u \). Position of \( \mathcal{F}_B \) coincident with the Center of Mass (CoM) \( p \). Rotation matrix mapping coordinates in \( \mathcal{F}_B \) to coordinates in \( \mathcal{F}_R \) \( R \). Configuration of the vehicle \( q \). Angular velocity of \( \mathcal{F}_B \) w.r.t. \( \mathcal{F}_W \) \( \omega \). Mass of the vehicle \( m \). Vehicle’s Inertial matrix w.r.t. \( \mathcal{F}_B \) expressed in \( \mathcal{F}_R \) \( J \). Control force applied at the CoM expressed in \( \mathcal{F}_B \) \( u_1 \). Control moment applied at the CoM expressed in \( \mathcal{F}_B \) \( u_2 \). Feasible set of the control force \( u \). Feasible set of the projection of \( u_1 \) on the xy plane in \( \mathcal{F}_B \) \( u_{xy} \). \( i \)-th vector of the canonical basis of \( \mathcal{F}_R \) with \( i = 1, 2, 3 \) \( e_i \).

Reference position for \( p \) at time \( t \) \( p_t \). Reference rotation matrix for \( R \) at time \( t \) \( R_t \).

\[ \mathcal{U}_{xy} = \{ u_1, u_2 \}^T \in \mathbb{R}^2 \],

where the \textbf{laterally bounding set} \( \mathcal{U}_{xy} \) is a set that contains the origin. We define \( \mathcal{U}_1 = \{ u_1 \in \mathbb{R}^3 \mid (u_1, u_2)^T \in \mathcal{U}_{xy}, u_3 \geq 0 \} \). Note that \( \mathcal{U}_{xy} \) can be constant or even be changing depending of \( u_3 \), as shown in Figure 1. The dynamics of the aerial platform is then

\[ m \ddot{r} = -mg e_z + Ru_1, \quad J \dot{\omega} = -\omega \times J \omega + u_2. \]

Some particularly relevant cases of LBF follow.

\textit{Case 1} (Underactuated aerial vehicle) When \( \mathcal{U}_{xy} = \{0\} \) the total force is always oriented as \( R e_3 \) and model (2) becomes the underactuated quadrorotor model considered in \cite{11, 13, 15}. \textit{Case 2} (Conic LBF) When \( \mathcal{U}_{xy} = \{ (u_1, u_2)^T \in \mathbb{R}^2 \mid u_1^2 + u_2^2 \leq (\tan \alpha)^2 u_3^2 \} \), model (2) approximates the case of hexarotors with tilted propellers \cite{5, 6, 9}, for which the set of allowable \( \mathcal{U}_1 \) forces has the pseudo-inverted-pyramidal shape. The quantity \( \alpha \) is a parameter that represents the tilting angle of the propellers (hexarotor).

\textit{Case 3} (Cylindrical LBF) When

\[ \mathcal{U}_{xy} = \{ (u_1, u_2)^T \in \mathbb{R}^2 \mid u_1^2 + u_2^2 \leq r_3^2 \}, \]

model (2) approximates the case of a multirotor with \( n_m \) main propellers pointing up and \( n_s \) secondary less powerful propellers tilted 90 degrees w.r.t. the main ones, like the one presented in \cite{17}, for which the set of allowable \( \mathcal{U}_1 \) forces can be approximated by a pseudo-cylindrical shape

\section{III. Full-Pose Geometric Control on SE(3)}

Let be given a full-pose trajectory \( \mathbf{q}_i(t) = (p_i(t), R_i(t)) : [t_0, t_f] \rightarrow \text{SE}(3) \), where \( p_i(t) \in \mathbb{R}^3 \) is the reference position trajectory and \( R_i(t) \in \text{SO}(3) \) is the reference attitude trajectory. Inverting (2), the nominal inputs to track \( q_i(t) \) are obtained as \( u^1_i = R_i^T (m \omega e_z + m \omega p_i) \) and \( u^2_i = \omega \times J \omega + J \dot{\omega} \), where \( \omega_i \) is defined by \( [\omega_i]_x = R_i^T \dot{R}_i \).

\textbf{Definition 1.} \( \mathbf{q}_i(t) \) is feasible if \( u^1_i(t) \in \mathcal{U}_1 \forall t \in [t_0, t_f] \).

Exact full-pose (6D) tracking is possible only if \( q_i(t) \) is feasible. However in real world it is not granted that \( q_i(t) \) will be such for the particular LBF platform in use. For this reason, we propose a controller that works (in the sense that the tracking of \( p_i(t) \) is still guaranteed and no singularity appears) even if \( q_i(t) \) is not feasible. Consider the position error \( e_p = p - p_i \), velocity error \( e_v = \dot{p} - \dot{p}_i \), and two positive definite gain matrices \( K_p \) and \( K_v \). Then consider

\[ f_i = m \ddot{p}_i + m \omega_3 (K_p e_p - K_v e_v), \]

representing the reference total control force that ideally one would like to apply to the aerial vehicle CoM if the system would be completely fully actuated, i.e., if \( \mathcal{U}_1 = \mathbb{R}^3 \).

The set of orientations that allow to apply \( f_i \) to the CoM of the LBF aerial vehicle is defined as \( \mathcal{T}(f_i) = \{ R \in \text{SO}(3) \mid R^T f_i \in \mathcal{U}_1 \} \). For an underactuated collinear multi-rotor system the set \( \mathcal{T}(f_i) \) is formed by any \( R \) such that \( R e_3 \) and \( f_i \) are parallel, i.e., \( R e_3 \times f_i = 0 \). For a generic LBF aerial vehicle the set \( \mathcal{T}(f_i) \) may contain also \( R \)’s for which \( R e_3 \times f_i \neq 0 \). Therefore we have the following.

\textbf{Proposition 1.} The set \( \mathcal{T}(f_i) \) is always nonempty \( \forall f_i \in \mathbb{R}^3 \).

\textbf{Proof.} If \( f_i \neq 0 \) then, by definition of \( \mathcal{T}(f_i) \) and \( \mathcal{U}_1 \), \( \mathcal{T}(f_i) \cap \{ R \in \text{SO}(3) \mid R e_3 = \frac{f_i}{\| f_i \|} \} \neq \emptyset \). If \( f_i = 0 \) then \( \mathcal{T}(f_i) = \text{SO}(3) \). \hfill \Box

The proposed controller exploits a cascaded structure by choosing, at each time \( t \), a desired orientation \( R_d \in \text{SO}(3) \) that belongs to \( \mathcal{T}(f_i) \) and also minimizes a given cost function w.r.t. \( R_d \). Then one can use the fully actuated rotational dynamics to track \( R_d \) and, in turn, track the reference position \( p_i \). If \( q_i \) is feasible then \( R_d \) will exponentially converge to \( R_d \). Otherwise, only the best feasible orientation will be obtained. Therefore the controller implicitly prioritizes the position trajectory tracking against the orientation one.

Define \( \mathcal{T}(f_i, R_c) \subset \mathcal{T}(f_i) \) as the set of rotation matrices that solve \( \min_{R_c \in \mathcal{T}(f_i)} J(R_c, R_d) \), where \( J : \text{SO}(3) \times \text{SO}(3) \rightarrow \mathbb{R} \geq 0 \) is an arbitrarily chosen cost function that represents the degree of similarity between \( R_c \) and \( R_d \) one is interested in. The elements in \( \mathcal{T}(f_i, R_c) \) represent orientations of the LBF that allow to apply \( f_i \) and minimize the function \( J \) w.r.t. \( R_c \).

Consider that, at each time \( t \) a desired orientation \( R_d \in \mathcal{T}(f_i, R_c) \) is chosen. Furthermore, whenever \( R_c \in \mathcal{T}(f_i, R_c) \) then \( R_d \) must be chosen equal to \( R_c \). Then define the rotation error \( e_R = \frac{1}{2} (R_d^T R - R_d^T R_d)^\top \), and the angular velocity error \( e_\omega = \dot{\omega} - R_d^T R_d \omega_d \) where \( \omega_d \) is the inverse map of \( \dot{x}|_x \) and \( \omega_d \) is the angular velocity associated to \( R_d \). Consider then the following control law

\[ u_1 = \text{sat}_{\omega_3} \left( (f_i^T R_c e_3) e_1 + (f_i^T R_c e_2) e_2 + (f_i^T R_c) e_3 \right) \]

\[ u_2 = \omega \times J \omega - K_R e_R - K_\omega e_\omega - J \left( \omega \right) R_d^T R_d \omega_d - R_d^T R_d \omega_d \]

where sat_{\omega_3}(x) is a vector in \( \mathcal{U}_{\omega_3} \) with the same direction of \( x \), that minimizes the distance from \( x \). \( K_R = k_R I \) and \( K_\omega = k_\omega I \) are the gain matrices with \( k_R > 0 \) and \( k_\omega > 0 \).

\textbf{2} Note that even if a cascaded structure is used, there is no time-scale separation assumption in the proposed controller.
Notice that only in the very special case in which \( \mathcal{M}_r = \{0\} \) the control in (4)-(5) resembles the one in [15] while its form is in general very different from it. Another difference with [15] is that the computation of \( \mathbf{R}_d \) in [15] is done from the position trajectory using the differential flatness property while here is computed ensuring input feasibility and minimizing \( J \) w.r.t. \( \mathbf{R}_r \).

In order to prove the convergence properties of the proposed controller let us consider the following error function between two rotation matrices \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \) to be \( d(\mathbf{R}_1, \mathbf{R}_2) = \frac{1}{2} \text{tr} \left( \mathbf{I} - \mathbf{R}_1^T \mathbf{R}_2 \right) \).

**Theorem 1.** Assume that \( \mathbf{R}_d(t) \in \mathcal{A}(f_r(t)) \) for any \( t \) and that \( \mathbf{ω}_d(t) \) and \( \mathbf{ω}_d(t) \) are well defined for any \( t \). Consider the control \( u_1 \) and \( u_2 \) defined at (4) and (5).

Assume that the initial condition satisfies
\[
d(\mathbf{R}(0), \mathbf{R}_d(0)) < 2,
\]
\[
\| \mathbf{e}_\omega(0) \|^2 < \frac{2}{\lambda_{\text{min}}(J)} k_R (1 - d(\mathbf{R}(0), \mathbf{R}_d(0)))
\]

Then, the zero equilibrium of the tracking errors \( \mathbf{e}_\mathbf{R}, \mathbf{e}_\omega, \mathbf{e}_p \) and \( \mathbf{e}_v \) is exponentially stable. The region of attraction is characterized by (6) and (7).

**Proof.** The proof is divided into two parts. We first show that, if the \( \mathbf{R}(0) \) and \( \mathbf{e}_\omega(0) \) satisfy, respectively, (6) and (7), then \( \mathbf{R}(t) \) converges exponentially to \( \mathbf{R}_d(t) \), in the sense that the function \( d(\mathbf{R}(t), \mathbf{R}_d(t)) \) goes exponentially to zero. Secondly, we characterize the transitive error dynamics and, based on the fact that \( \mathbf{R}(t) \) converges exponentially to \( \mathbf{R}_d(t) \), we show that also \( \mathbf{e}_p \) and \( \mathbf{e}_v \) goes exponentially to zero. The time derivative of \( \mathbf{e}_\omega \) is
\[
J \mathbf{e}_\omega = J \mathbf{v} + J \left( \left[ \mathbf{ω}_d \right] \times \mathbf{R}_d^T \mathbf{R}_d \mathbf{ω}_d - \mathbf{R}_d^T \mathbf{R}_d \mathbf{ω}_d \right).
\]

Plugging the rotational part of (2) into (8) and substituting \( \mathbf{u}_2 \) from (5), we get \( J \mathbf{e}_\omega = -k_\mathbf{R}\mathbf{e}_\mathbf{R} - k_{\mathbf{ω}}\mathbf{e}_\omega \). In [15], it is shown by exhibiting a suitable Lyapunov function, that under conditions in (6) and in (7), the zero equilibrium of the attitude tracking error \( \mathbf{e}_\mathbf{R}, \mathbf{e}_\omega \) is exponentially stable and that there exist two positive constants \( \alpha, \beta \) such that
\[
d(\mathbf{R}(t), \mathbf{R}_d(t)) < \alpha e^{-\beta t} d(\mathbf{R}(0), \mathbf{R}_d(0)).
\]

We determine now the error dynamics of the translational part. Substituting \( \mathbf{u}_1 \) from (4) in the translational part of (2) we obtain
\[
\dot{\mathbf{m}} = -m \mathbf{g} + \mathbf{f}_r + \mathbf{R}_1 \mathbf{f}_r - \mathbf{K}_p \mathbf{e}_p - \mathbf{K}_v \mathbf{e}_v + \mathbf{γ},
\]
where \( \mathbf{γ} = \mathbf{R}_1 \mathbf{f}_r - \mathbf{f}_r \). It easily follows that
\[
\dot{\mathbf{m}} \mathbf{e}_v = -\mathbf{K}_p \mathbf{e}_p - \mathbf{K}_v \mathbf{e}_v + \mathbf{γ}.
\]

Consider (9) and observe that, since \( \mathbf{R}_d \in \mathcal{A}(f_r) \) for any \( t \), we have that there exist two positive constants \( C, \rho \) such that
\[
\| \mathbf{γ}(t) \| \leq Ce^{-\rho t} \| \mathbf{γ}(0) \|.
\]

Let \( \mathbf{x} = [\mathbf{e}_v \mathbf{e}_p]^T \) then, (10) can be written in vector form as
\[
\mathbf{x} = \mathbf{A} \mathbf{x} + \mathbf{B} \gamma
\]

\[
A = \frac{1}{m} \begin{bmatrix} -\mathbf{K}_p & -\mathbf{K}_v \\ \mathbf{I} & 0 \end{bmatrix}, \quad B = \frac{1}{m} \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}.
\]

Since \( \mathbf{K}_p, \mathbf{K}_v \) are both positive definite matrices, we have that \( \mathbf{A} \) is a Hurwitz matrix.

Observe that (11) is the cascade of a linear stable system and an exponential stable signal. Then, the statement of the Theorem follows from a technical refinement of Lemma 4.7 in [20], i.e., the cascade of a asymptotically stable linear system and of an exponentially stable system is exponentially stable.

\[\square\]

A block diagram that shows the main subsystems of the proposed control architecture is provided in Fig. 2.

**Theorem 1.** Assume that \( \mathbf{q}_d(t) \) is feasible and satisfies the property that \( \mathbf{u}_1^* \) is sufficiently inside \( \mathcal{M}_1 \), meaning that there exists a time instant \( \bar{t} \) and a positive number \( \varepsilon \) such that the distance of \( \mathbf{u}_1^* \) from the boundary of \( \mathcal{M}_1 \) is greater than \( \varepsilon > 0 \) for all \( t > \bar{t} \), i.e.,
\[
\text{dist}(\mathbf{u}_1^*(t), \partial \mathcal{M}_1) > \varepsilon, \quad \forall t > \bar{t}.
\]

**Corollary 1.** Assume \( \mathbf{q}_d(t) \) is a feasible trajectory and that it satisfies the additional property in (12). Assume that \( \mathbf{R}_d(t) \in \mathcal{A}(f_r(t)) \) for any \( t \) and that \( \mathbf{ω}_d(t) \) and \( \mathbf{ω}_d(t) \) are well defined for any \( t \). Consider the control \( u_1 \) and \( u_2 \) defined at (4) and (5). Assume that the initial condition satisfies (6) and (7). Then the zero equilibrium of the tracking errors \( \mathbf{e}_\mathbf{R}, \mathbf{e}_\omega, \mathbf{e}_p \) and \( \mathbf{e}_v \) is exponentially stable and there exists a time instant \( \bar{t} \geq t_0 \) such that \( \mathbf{e}_\mathbf{R}(t) = \mathbf{e}_\omega(t) = \mathbf{e}_p(t) = \mathbf{e}_v(t) = 0 \) for all \( t > \bar{t} \). The region of attraction is characterized by (6) and (7).

**Remark 1.** The proposed controller (in particular the attitude controller (5)) relies on the availability of \( \mathbf{ω}_d \) and \( \mathbf{ω}_d \). These quantities depend in turn on \( \mathbf{R}_d \) which is the output of an optimization algorithm executed at each control step. In order for \( \mathbf{ω}_d \) and \( \mathbf{ω}_d \) to be well defined and available the optimization must ensure a sufficient smoothness of \( \mathbf{R}_d \). This could be enforced by adding, e.g., a regularization term in the cost function \( J \). If in the real case at hand this is not possible (or not implementable), then at each time instant in

---

Preprint version final version at http://ieeexplore.ieee.org/

Accepted for IEEE Transaction on Robotics 2018
which \( R_d \) is not smooth the attitude controller will undergo a new transient phase. In practice, see Sec. V, we have experimentally ascertain that the presence of a few isolated non-smooth instants does not constitute at all a real problem for the stability of the implementation and that regularization is actually not needed for practical stabilization.

IV. COMPUTATION OF \( R_d \) FOR AN IMPORTANT CASE

The proposed control method is kept on purpose general regarding two main features: the choice of \( \mathcal{U}_{xy} \) in (1) and the choice of the cost function \( J \). The former allows the method to be used for a large set of aerial vehicles with different actuation capabilities. The latter allows the engineer to customize the definition of similarity between two orientations in order to comply with the particular task at hand. In this section we illustrate how these two general features are particularized for a specific meaningful case.

In particular, we consider the case of \( \mathcal{U}_{xy} \) defined in (3) and the following choice of cost function \( J \)

\[
J(R, R') = 1 - b_{2r}^T b_3',
\]

(13)

where \( R_r = \{b_{1r}, b_{2r}, b_{3r}\} \) and \( R' = \{b_1', b_2', b_3\} \). The cost function \( J \) in (13) is minimized whenever \( b_{2r} = b_{2}' \) and maximized whenever \( b_{2r} = -b_{2}' \). In the following we show how it is possible to efficiently compute an \( R_d \) that belongs to \( \mathcal{R}(f, R_r) \) and is also equal to \( R \), if \( R_r \in \mathcal{R}(f, R_r) \). These are in fact the requirements needed for \( R_d \) in order for Theorems 1 and Corollary 1 to be valid.

Let us start by instantiating \( \mathcal{R}(f_r) \) for this particular case. From simple geometrical considerations on the cylindrical shape of the set \( \mathcal{U} \), it is easy to see that

\[
\mathcal{R}(f_r) = \left\{ R' \in SO(3) \mid f_r^T b_3' \geq \sqrt{\|f_r\|^2 - r_{xy}^2} \right\},
\]

(14)

which states that the vector \( f_r \) must lie within the cylinder of radius \( r_{xy} \) generated about the axis \( b_3' \).

Using (13) and (14) we can rewrite the minimization problem in terms of the only variable \( b_3' \), instead of the whole matrix \( R' \), as

\[
\min_{f_r^T b_3' \geq \sqrt{\|f_r\|^2 - r_{xy}^2}} -b_{2r}^T b_3',
\]

(15)

where \( r_{xy}, f_r^T \) and \( b_{2r}' \) are the given values of the problem.

In the case that \( f_r^T b_{3r} \geq \sqrt{\|f_r\|^2 - r_{xy}^2} \) then \( b_{3}' = b_{3r} \) is the solution to (15). Otherwise, let us write \( b_3' \) as the sum of two components \( b_3' = b_{31}' + b_{32}' \), where \( b_{31}' \) is parallel to the plane spanned by \( b_{3r} \) and \( f_r \), while \( b_{32}' \) is perpendicular to it, i.e., is parallel to \( b_{3r} \times f_r \). It is easy to see that the cost function in (15), i.e., \( b_{31}' b_3' \), is not affected by \( b_{31}' \), in fact \( b_{3}^T b_3' = b_{31}' b_{31}' + b_{32}' b_{32}' = 0 + b_{32}' b_{32}' \). The vector \( b_{32}' \) can be written using the Rodrigues’ rotation formula as \( b_{32}'(\theta) = b_{3r} \cos \theta + (k \times b_{3r}) \sin \theta + k (k \cdot b_{3r}) (1 - \cos \theta) \), where \( k = \frac{b_{3r} \times f_r}{\|b_{3r} \times f_r\|} \) and \( \theta \) is the rotation angle that univocally defines \( b_{32}' \). Noting that the constraint \( \|b_{3}'\|^2 = 1 \) is automatically verified by \( b_{32}'(\theta) \)

for any \( \theta \), we further simplify (15) in terms of the only scalar variable \( \theta \) as

\[
\min_{f_r^T b_3' \geq \sqrt{\|f_r\|^2 - r_{xy}^2}} -b_{2r}^T b_{32}'(\theta).
\]

(16)

Given that the feasible set is connected, the minimization problem (16) can be efficiently solved numerically using a bisection method (as the one shown in Algorithm 1).

In order to finally compute \( R_d \) from \( b_{3d} \) we suggest to employ the following formula, as done in [15]: \( R_d = [b_{3d} \times b_{3r}] \times b_{3d} \). Finally, we note that if \( R_r \in \mathcal{R}(f, R_r) \), then \( f_r^T b_{3d} \geq \sqrt{\|f_r\|^2 - r_{xy}^2} \), which, as we previously stated, implies that \( b_{3d} = b_{3r} \), and then results \( R_d = R_r \), as asked by the second requirement on the computation of \( R_d \).

The described algorithm takes a negligible time to be run on a standard computer, thus allowing a real time numerical control implementation at frequencies that are way below 1 ms for each control loop, if needed by the application. In case of different sets \( \mathcal{U}_{xy} \) and different cost functions \( J \) either similar efficient approaches can be used or the method presented here can be used as a conservative approximation.

V. EXPERIMENTS

1) Experimental Setup: The LBF platform used to perform the experiments is the Tilt-Hex robot, an in-house developed fully actuated vehicle (see Fig. 4). The Tilt-Hex is an example of a platform that can be controlled by the full-pose geometric controller presented in Sec. III. We carefully chose the Tilt-Hex to perform the experiments as the platform is able to emulate different values of \( r_{xy} \in [0, r_{xy_{\text{max}}}] \) (see (3)), which means that we can emulate seamlessly underactuated and fully actuated platforms. Furthermore, the Tilt-Hex can emulate for validation purposes the cylindric force constraint of (3) by choosing \( r_{xy} \) small enough and avoiding trajectories that require extreme values of \( u_3 \), see Fig. 3.

All components of the Tilt-Hex are off-the-shelf available or 3D printable. The diameter of the Tilt-Hex (distance between two propeller hubs) is 0.8m. The total mass, including a 2200 mAh battery, is \( m = 1.8 \)kg. The propellers are tilted first about the axis that connects the rotor with the center of the hexator shape (\( \alpha = \pm 35^\circ \)), then about the axis that is perpendicular to the previous rotation axis and \( e_3 (\beta = 25^\circ) \). These angles ensure a balanced selection between large lateral forces and inefficient losses as a result of internal forces.

Algorithm 1: Bisection used to solve problem (16)

Data: \( n \) (number of iterations \( \propto \) solution accuracy)
Data: \( b_{3r}, f_r \), and \( r_{xy} \) (gives of the problem)
1 \( k \leftarrow \frac{b_{3r} \times f_r}{\|b_{3r} \times f_r\|}, \theta_{\text{max}} \leftarrow \arcsin(\|k\|), t \theta \leftarrow \theta_{\text{max}}/2 \)
2 for \( i = 1 \) to \( n \) do
3 \( \text{if } f_r^T b_3'(\theta) \geq \sqrt{\|f_r\|^2 - r_{xy}^2} \text{ then } \theta \leftarrow \theta - \frac{1}{2} \theta_{\text{max}} \)
4 \( \text{else } \theta \leftarrow \theta + \frac{1}{2} \theta_{\text{max}} \)
5 return \( \theta \)
The 30.5 cm propeller blades are powered by MK3638 brushless motors provided by MikroKopter. The maximum lift force of a single motor-propeller combination is 12 N. The electronic speed controller (ESC) driving the motor is a BL-CTRL-2.0 from MikroKopter. The controller, running on the ESC, is an in-house development [21] that controls the propeller rotational speed in closed loop at a variable frequency (e.g., when the propeller rotation speed is 70 Hz the control frequency is 3.29 kHz). An inertial measurement unit (IMU) provides accelerometer and gyroscope measurements (500 Hz) and a marker-based motion capture system provides position and orientation measurements (100 Hz) of the platform. The motion capture and IMU measurements are fused via a UKF state estimator to obtain the full state at 500 Hz.

In the real implementation we have omitted, on purpose, all the terms in which $\omega_d$ and $\omega_d$ appear in the control (or equivalently, we have considered as if, virtually, $\omega_d = 0$). We have done so in order to assess the level of practical importance of those terms and how the controller is in practice robust to the non-correct evaluation of these terms. As it will be seen next it turned out that the tracking performance is still very good even without those terms, therefore one can still obtain acceptable result by neglecting them. However, as requested from the theory and also seen in preliminary simulations which we performed but are not reported in the paper for lack of space and also because much less significant than real experiments, a perfect tracking is guaranteed only if those terms are considered.

Finally, the controller has been implemented in Matlab-Simulink and runs at 500 Hz on a stationary base PC connected to the Tilt-Hex through a serial cable. As the computational effort of the controller is very low (considerably below 1 ms per control loop) it could be ported easily to an on-board system. Based on our experience with a similar porting, we expect the performances of an onboard implementation to be much better than the Matlab-Simulink implementation, thanks to the possibility of reaching a faster control frequency (larger than 1 kHz) and almost real-time capabilities (latency below 1 ms). The presence of the cable is clearly disturbing for the tracking purpose since it produces an unmodeled wrench. Furthermore, the oscillations of the cable are rather erratic and therefore hard to model and compensate. Therefore, all considered, the experiments proposed in Section V-2 represent a worst case scenario from this point of view, in the sense that an onboard implementation can only perform better than what we tested.

2) Experimental Validations: We conducted three experimental validation campaigns, in which the tasks has been always to follow a given reference $q(t) = (p(t), \dot{p}(t), R(t), \dot{R}(t))$. In the Experimental Batch 1 the value of $r_{xy}$ in (3) and of $\mathbf{R}$, is kept constant and three experiments are perfomed: Exp.1.1, Exp.1.2 and Exp.1.3, which are detailed in the following. In the Experiment 2, $\mathbf{R}$ varies over time. Finally, in the Experiment 3 is $r_{xy}$ that varies over time.

For the reader’s visual convenience the rotation matrices used internally by the controller, have been converted in the plots to roll-pitch-yaw angles, with the convention $\mathbf{R} \rightarrow \phi, \theta, \psi$. In plots where data are very noisy a filtered version (darker color) is presented together with the original data (lighter color in background). We shall refer to the plots in a figure directly by the name of plotted signal, which is easily understandable from the legends present in each plot.

The interested reader is referred to the multimedia attachment of this paper to fully enjoy the videos all the experiments. Furthermore, all the experimental data (with suitable scripts to plot them) are provided for download at the following link http://homepages.laas.fr/afranchi/files/2017/dataset1.zip.

3) Experimental Batch 1: In this batch we impose $p(t) = 0$ and $p(t) = 1$ m, while $p(t)$ oscillates sinusoidally between $-1.2$ m and 1.2 m with time-varying frequency, chosen such that the envelope of $p(t)$ is first quasi-linearly increasing from 0 m/s$^2$ up to 5.9 m/s$^2$ and then quasi-linearly decreasing down to 0 m/s$^2$ – see the corresponding signals in Fig. 5. On the other side we set $\mathbf{R}(t) = I_{3\times3}$.

In Exp.1.1 a value of $r_{xy} = 3$ N has been selected, which fits well inside the actual maximum lateral force of the Tilt-Hex given its mass of $m = 1.8$ kg. This means that in the parts of the trajectory in which $|\ddot{p}| > \frac{3N}{1.8kg} = 1.66$ m/s$^2$ we expect the controller to let the platform deviate from $\mathbf{R}$, in order to track the high lateral acceleration. On the other side we expect a good independent tracking of position and orientation when $\ddot{p} \leq 1.66$ m/s$^2$. In fact, Exp.1.1 is meant to illustrate the canonical behavior of proposed controller when controlling a fully-actuated LBF platform.
In Exp. 1.2 we tell the controller that $r_{xy} = 0$ N (underactuated aerial vehicle). Therefore we expect the controller to let the platform deviate almost always from $R_p$. This experiment is meant to show that the proposed controller can handle the (classic) underactuated case, thus not requiring the user to switch between different controllers.

In Exp. 1.3 we replace our controller with the state-of-the-art controller presented in [6], a controller that does not take into account the input saturations. This experiment is meant to show how the proposed controller outperforms the state of the art controllers in terms of robustness and stability.

**Experiment 1.1:** Fig. 5 visualizes the main results of Exp 1.1. There are three clearly distinct temporal phases separated by the vertical dashed lines in the plots and defined by $t \in T_1 = [0s, 25.8s]$, $t \in T_2 = [25.8s, 74s]$, and $t \in T_3 = [74s, 95s]$, respectively.

In the first and third phases $q_r(t)$ is always feasible. In fact $|\dot{p}_x|$ is always below $1.66 m/s^2$, see discussion above. In the second phase, instead, $q_r(t)$ is not always feasible and in the middle of the second phase, in the neighborhood of $t = 49s$, $q_r(t)$ is mostly unfeasible, since $|\dot{p}_x|$ has peaks of $5.9 m/s^2$ (3.5 times the maximum lateral acceleration attainable while keeping a horizontal orientation).

Accordingly to what expected, in the ‘feasible’ phases ($T_1$ and $T_3$) both the orientation and position tracking errors w.r.t. $q_r(t)$, are relatively low. In particular we have $\|p(t) - p_r(t)\| < 0.02 m$ and $|\theta(t) - \theta_r(t)| < 1.7^\circ$ and zero average for the two errors. In the ‘unfeasible’ phase ($T_2$), the position tracking is still good ($\|p(t) - p_r(t)\| < 0.06 m$) while $R_p(t)$ sensibly deviates from $R_p(t)$ with a peak overshooting $20^\circ$ for $|\theta_r(t) - \theta_r(t)|$. In fact, tilting is the only way by which the platform can track the desired position, given the lateral force bounds. It is interesting to note that not only the reference position is well tracked along the whole experiment, but also the translational velocity and acceleration are.

One can also appreciate how the controller keeps always the lateral force $u_{1,2}$ within the requested bounds and at the same time touches and stays on the bounds several times for several seconds. This is a clear index that the controller exploits at best the platform capabilities. Comparing the plots one can notice also how when the lateral force is saturated, the controller exploits the platform tilting in order to compensate for the partial loss of control authority and attain the force required to produce the needed acceleration.

Figure 4 shows the Tilt-Hex performing Exp. 1.1 in three different time instants. Furthermore, we encourage the reader to watch the multimedia attachment showing this and the other experiments. Finally, for the sake of completeness, we present also the actual six rotor spinning frequencies $w_1, \ldots, w_6$.

**Experiment 1.2:** To test the behavior of the controller with an underactuated aerial system, in Exp. 1.2 we set $r_{xy} = 0$ N and let the controller track the same trajectory of Exp. 1.1. In this way the Tilt-Hex should behave like a collinear, underactuated, multi-rotor. Figure 6 shows the main plots, while the plots that are similar to the ones in Fig. 5 are omitted.

Contrarily to Exp. 1.1, in Exp. 1.2, phases $T_1$ and $T_2$ do not exist. The whole experiment is a long unfeasible phase due to the constraint $r_{xy} = 0$ N, which makes impossible, at any time, to track the constant $R_p(t) = I_{3 \times 3}$ while following the sinusoidal reference position trajectory. The orientation tracking of Exp. 1.2 is compared to the one Exp. 1.1. In the period of time $T_1$ defined for Exp. 1.1, $|\theta(t) - \theta_r(t)|$ in Exp. 1.2, reaches $11^\circ$, i.e., 6.5 times the peak of Exp. 1.1 in the same period. In the period of time $T_2$, $|\theta(t) - \theta_r(t)|$ in Exp. 1.2, reaches $31^\circ$ i.e., about 1.5 times the peak of Exp. 1.1 in the same period. Regarding the translational behavior, the peak of the position tracking error is about 3 times larger (in the period $T_1$) and 1.4 times larger (in the period $T_2$), when compared to the error peak of Exp. 1.1 in the same periods. This is due to the fact that full actuation helps in minimizing the position tracking error too. Furthermore, we can see that
that the inputs \( u_1 \) and \( u_2 \) remain zero as expected during the full trajectory tracking, as required. Finally, for completeness, we present also in this case the six rotor spinning frequencies \( w_1, \ldots, w_6 \).

Experiment 1.3: In order to compare with the state-of-the-art methods such as [6], in Exp. 1.3 we tested the controller with a saturated rotor spinning velocity with the minimum and maximum values in Exp. 1.1 (43 Hz \( \leq w_i \leq 83 \text{ Hz}, \ i \in [1 \ldots 6] \)). The results are depicted in Fig. 7. The platform tracks well the reference trajectory till the input reaches its limit (\( t=34 \text{ s} \)) (see the dashed horizontal lines in the plot of \( w_1, \ldots, w_6 \)). After that time the controller asks for spinning velocities outside the limits, which are hardly saturated. The trajectory tracking performance decreases rapidly, until the system becomes completely unstable. The resulting lateral force bound changes over time, to stress the fact that the presented controller can seamlessly under and fully-actuated platforms and moreover with platforms that can actively change between these two configurations during flight, as the one presented in [9]. The plots of are reported in Fig. 9.

The steady state \( \mathbf{p}_r(t) \) consists out of two regular sinusoidal motions along the \( \mathbf{x}_W \) and \( \mathbf{y}_W \) axes with an amplitude of 1.3 m and 0.5 m, respectively, and constant frequencies, while \( \mathbf{R}_r(t) = \mathbf{I}_{3 \times 3} \) is constant. The lateral force bound \( r_{xy} \) is changed over time, in particular, it is \( r_{xy}(t) = 0 \text{ N} \) for \( t \in [0 \text{s}, 18 \text{s}] \), \( r_{xy}(t) = 10 \text{ N} \) for \( t \in [38 \text{s}, 56 \text{s}] \), and it is linearly increasing from 0 N to 10 N for \( t \in [18 \text{s}, 38 \text{s}] \).

As it should be, the position tracking is always good. However, initially, the system cannot track at the same time the reference position and the reference orientation. Until \( t = 34 \text{s} \) the system is partially fully actuated (w.r.t. the trajectory to be followed). The orientation tracking gradually improves. At \( t = 34 \text{s}, r_{xy} \) is large enough to track the reference orientation at any time. The behavior of \( r_{xy} \) is visualized with the dashed lines in the plot of \( u_1, u_2 \), which are always kept within the bounds. The fact that the lateral force bound changes over time does not deteriorate the behavior of the controller, which is instead able to cope with the time-varying constraint exploiting the platform capability always at its best.
Finally, notice how the ranges of the propeller spinning frequencies utilized by the controller $(w_1 \ldots w_6)$ naturally increase with the increase of $r_{xy}(t)$ at the benefit of completely tracking the full pose reference trajectory.

VI. CONCLUSIONS

In this paper we introduced the new class of Laterally-Bounded Force (LBF) aerial vehicles. This class is general enough to encompass a large variety of recently conceived aerial vehicles having the possibility to actuate the thrust in a direction other than the principal one. Common underactuated platforms are included in this class as a degenerate (but fully admissible) case. For this class of vehicles we proposed a geometric controller in SE(3) that is able to let it track any feasible full-pose (6D) trajectory. The controller adapts seamlessly to the case the trajectory is (or becomes) not feasible or that the platform is (or becomes) underactuated. Being defined in SE(3) the controller is not prone to the singularities of local chart orientation representations. The practicability of the theory has been shown in real experiments. Furthermore this controller has been already used as inner loop controller for other projects involving also aerial physical interaction, as, e.g., in [10].

In future we plan to study adaptive and robust techniques to deal with parameter uncertainties and malfunction of some of the actuators.

REFERENCES


[18] G. Ducard and M.-D. Hua, “Discussion and practical aspects on control allocation for a multi-rotor helicopter,” in 1st Int. Conf. on UAVs in Geomatics, Zurich, Switzerland, Sep. 2011, pp. 95–100.

