Lyapunov stability of a coupled ordinary differential system and a string equation with polytopic uncertainties

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Chapter 1
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Abstract This chapter deals about the robust stability analysis of a coupled system made up of an uncertain ordinary differential system and a string equation. The main result states the robust exponential stability of this interconnected system subject to polytopic uncertainties. The Lyapunov theory transforms the stability analysis into the resolution of a set of linear matrix inequalities. They are obtained using projections of the infinite dimensional state onto the orthogonal basis of Legendre polynomials. The special structure of these inequalities is used to derive robust stability results. An example synthesizes the two main contributions of this chapter: an extended stability result and a robustness analysis. The example shows the efficiency of the proposed method.

1.1 Introduction

Infinite dimensional systems are natural systems which arise when a delay or a partial differential equation appears unavoidably in the modeling phase. In particular, vibrations of structure [9], torsion [8, 11] or heat propagation [3, 21] are examples of phenomena giving rise to infinite dimensional systems. The infinite dimension behavior is characterized by an infinite dimension state, belonging to a functional space. This consideration seems rather far from a natural modeling and its utility can be questioned.

That is why, as a first step, the infinite dimensional system can be approximated as a high but finite dimension model. Then, applying classical tools leads to the design of controllers. Nevertheless, it has been shown in [4, 18], that this truncation brings instability because of the well-known spill-over effect. Indeed, high frequen-

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cies can be exited by the controller and as they are not regulated, they can provoke a divergence of the solution.

Then, Distributed Parameters Systems (DPS) can be the most appropriated model for some physical systems and a truncation is not always possible. To overcome this problem, special tools have been developed for DPS systems. Without being exhaustive, we can note the work of Krstic and al. (see for instance [5, 14, 15]) using the backstepping methodology or the use of the semi-group theory [17, 22]. Considering a physical modeling, uncertainties of the parameters have to be taken into account to design efficient and robust control laws.

The work conducted here aims at using the second methodology to prove the robust exponential stability of an ordinary differential system coupled with a string equation. One of the main challenge in using semi-group theory consists in finding a scalar product for which the system is dissipative. In [1], we proposed a Lyapunov functional which encompasses the traditional notion of energy as it results from an optimization and is problem-oriented. This framework enabled us to transform the dissipativity proof for the coupled system into the feasibility of Linear Matrix Inequalities (LMI). We propose here to enlarge this study by considering less restrictive LMIs and to the case of uncertainties on the ordinary differential part of the interconnected system. This last part is the main contribution as it is commonly the first extension proposed when using a Lyapunov functional and as it requires a transformation of the LMI to get the robust stability result.

Section 2 is the problem statement, describing the coupled system, its main characteristics and the slight changes in the framework to deal with polytopic uncertainties. Then, after a brief reminder of some useful lemmas in Section 3,an extended exponential stability result is provided in Section 4. The last subsection of Section 3 is dedicated to the robustness analysis by taking advantage of the convexity of the LMIs with respect to the system's matrices. Finally, in Section 5, an example with stability pockets is studied.

Notations: In this paper, Ω is the closed set $[0,1]$ and $\mathbb{R}^+ = [0,+\infty)$. Then, $(x,t) \mapsto u(x,t)$ is a multi-variable function from $\Omega \times \mathbb{R}^+$ to $\mathbb{R}$. The notation u_t stands for $\frac{\partial u}{\partial t}$. We also use the notations $L^2 = L^2(\Omega;\mathbb{R})$ and for the Sobolov spaces: $\mathscr{H}^n = \{z \in L^2; \forall m \leqslant n, \frac{\partial^m z}{\partial x^m} \in L^2\}$. The dot product in L^2 is defined for $f,g \in L^2$ as $\langle f,g \rangle = \int_\Omega f(x)g(x)dx$. Then the norm associated to this scalar product is $\|f\|^2 = \langle f,f \rangle = \int_\Omega |f(x)|^2 dx$.
For any square matrices A and B, the operations 'He' and 'diag' are defined as follow: $\mathrm{He}(A) = A + A^\top$ and $\mathrm{diag}(A,B) = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. A positive definite matrix $P \in \mathbb{R}^{n\times n}$ belongs to the set $\mathbb{S}^n_+$ or more simply $P \succ 0$.

1.2 Problem statement

The coupled ODE/PDE system under consideration is the following one:

$$\begin{aligned}
\dot{X}(t) &= AX(t) + Bu(1,t), && t \geqslant 0, && (1.1a)\\
u_{tt}(x,t) &= c^2 u_{xx}(x,t), && x \in \Omega, t \geqslant 0, && (1.1b)\\
u(0,t) &= KX(t), && t \geqslant 0, && (1.1c)\\
u_x(1,t) &= -c_0 u_t(1,t), && t \geqslant 0, && (1.1d)\\
u(x,0) &= u^0(x),\ u_t(x,0) = v^0(x), && x \in \Omega, && (1.1e)\\
X(0) &= X^0, && && (1.1f)
\end{aligned}$$

where equation (1.1a) is an Ordinary Differential Equation (ODE), $X \in \mathbb{R}^n$ with $n \geqslant 1$, A and B are not necessary known matrices of appropriate dimensions. Equation (1.1b) is a Partial Derivative Equation (PDE), and more specifically, a string equation which is a one dimensional wave equation with a speed $c > 0$. x is the spatial variable and belongs to Ω and t is the time. Equations (1.1c) and (1.1d) are the boundary conditions at $x = 0$ and $x = 1$ respectively. The first boundary condition is of Dirichlet kind, so acting on $u(0,t)$, contrary to the second one, which is of Neumann type (i.e. acting on $u_x(1,t)$) and describes a boundary damping condition. This means that, for $c_0 > 0$, the string equation is damped (see [16] for instance). Equations (1.1e) and (1.1f) represent the initial conditions of the system, which are said to be compatible with the boundary conditions if they satisfy equations (1.1c) and (1.1d).

Remark 1. To ease the reading the variable t will be omitted when it is not needed.

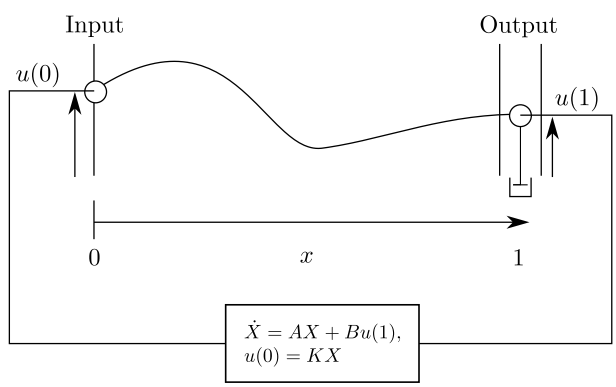


Fig. 1.1: Scheme of system (1.1).

This kind of systems are approximations of what arises when it comes to deal with vibrations in structure [9] or torsion in a drilling pipe for example (see [5, 8, 19]). The ODE (1.1a) is then the finite dimension controller acting on the top of the pipe. The measure of the angle at the bottom is the input of the ODE. That makes the schematic in Figure 1.1.

This system can be viewed in different manners. First, as it can be noticed in Figure 1.1, the PDE can be seen as a communication channel modeled by a string equation. Then, the signal u is the conveyed KX at $x = 0$ there is the input signal while at $x = 1$, this is the output. The communication channel is then stable ($c_0 > 0$) and the ODE is regulated thanks to the string equation. This is the interconnection between a stable PDE plant (i.e. $c_0 > 0$) and a possibly unstable ODE. Then, three scenari come out:

1. If A is Hurwitz, this communication channel is a perturbation and the stability analysis of the coupled ODE/PDE system is nothing more than a robustness analysis subject to this communication channel;
2. If A is not Hurwitz but $A+BK$ is, then the string equation is needed to stabilize the system and is not a perturbation anymore, it helps stabilizing;
3. Finally, if neither A nor $A+BK$ are Hurwitz, the stability of the coupled system relies mostly on the intrasec properties of the string equation.

This system under the three cases discussed previously has already been studied in [1] and a stability criteria has been derived in terms of LMIs. Several conclusions have been drawn and particularly, c_0 must be positive to get some results using this methodology. This means that the communication channel must be stable. Moreover, the obtained conditions are dependent on a perfect knowledge of all the parameters A, B, K, c and c_0.

To enlarge the result obtained in Theorem 2 of [1], a robustness analysis on the parameters A and B is proposed. The main assumption is that A and B are subject to polytopic uncertainties, that means the following holds:

$$[\,A\ B\,] \in \mathrm{Co}_{i=1,\dots m}\left\{[\,A_i\ B_i\,]\right\}, \tag{1.2}$$

where $m \in \mathbb{N}$, and the matrices A_i and B_i for $i = 1, \dots, m$ are known and constant. The notation "Co" means that the matrix $[\,A\ B\,]$ belongs to a convex set defined by the vertices $[\,A_i\ B_i\,]$. In other words, there exist weighting scalar functions λ_i, for $i = 1, \dots, m$ that maps $\mathbb{R}^+$ to $[0,\ 1]$, and such that for all $t \geqslant 0$, $\sum_{i=1}^m \lambda_i(t) = 1$ and

$$[\,A\ B\,] = \sum_{i=1}^{m} \lambda_i(t)\,[\,A_i\ B_i\,]. \tag{1.3}$$

This kind of uncertainty is commonly used in the LMI formulation because thanks to a convexity argument the robust problem can be easily reformulated in terms of higher complexity LMI. The number of edges to define the polytope is directly related to the complexity of the new LMIs and can then dramatically increase the computational burden.

As the method presented in [1] can also be computationally demanding, after a brief summary of the main lemmas used here, an extension is proposed to decrease the conservatism, keeping the same number of decision variables.

1.3 Preliminaries

In this preliminary part, we recall two main results used in the sequel.

1.3.1 Legendre polynomials and Bessel-Legendre inequality

To transform integral terms into an LMI formulation, we need to use integral inequalities. Based on the work in [20], the Bessel-Legendre inequality is reminded below:

Lemma 1 (Bessel-Legendre Inequality). *For any orthogonal family $(e_k)_{k\in\mathbb{N}}$ of L^2 with respect to the scalar product $\langle\cdot,\cdot\rangle$ and any function $f \in L^2$, the following inequality holds for $N \in \mathbb{N}$:*

$$\|f\|^2 \geqslant \sum_{k=0}^{N} \langle f, e_k\rangle \|e_k\|^{-2}.$$

This inequality becomes an equality when N goes to infinity. This last property makes us think that as N increases, the conservatism induced by this inequality can be reduced as much as desired.
We choose to apply this result to the orthogonal family of Legendre polynomials $(\mathscr{L}_k)_{k\in\mathbb{N}}$ for several reasons. First, because it is a polynomial and consequently, its derivative can be expressed in strictly lower order polynomials. Secondly, because the boundary conditions are simple and make calculations easier. The definition of Legendre polynomials is reminded below:

$$\forall k \in \mathbb{N},\ \forall x \in [0,1], \quad \mathscr{L}_k(x) = (-1)^k \sum_{l=0}^{k} (-1)^l \binom{k}{l} \binom{k+l}{l} x^l,$$

with $\binom{k}{l} = \frac{k!}{l!(k-l)!}$. For more properties about this polynomials, the reader can refer to [1, 10, 20].

1.3.2 Convexity Lemma

This second lemma is useful when it comes to convexify an LMI. This is particularly useful to derive a robustness criterion. This lemma is taken from [6, 7].

Lemma 2. *For some given matrices $\mathscr{M} = \mathscr{M}^\top, \mathscr{Y} = \mathscr{Y}^\top$ and $\mathscr{Z}$ of appropriate dimensions, the following statements are equivalent:*

1. *The matrix inequality $\mathscr{M} - \mathscr{Z}^\top \mathscr{Y} \mathscr{Z} \prec 0$ holds.*

2. *There exists a matrix $\mathscr{X}$ and a scalar $\mu > 0$ such that the following matrix inequality holds*

$$\mathscr{M} + \mathrm{He}(\mathscr{X}^\top \mathscr{Z}) + \mathscr{X}^\top (\mu I + \mathscr{Y})^{-1} \mathscr{X} + \mu \mathscr{Z}^\top \mathscr{Z} \prec 0,$$

together with $\mu I + \mathscr{Y} \succ 0$.

1.4 From stability to robust stability analysis

The desired stability property here is the well-known exponential stability for (1.1) with A, B, K and c known. To define it correctly, we first need to know in which space the solutions of system (1.1) belong to. This is related to the well-posedness of this system. These results are quite technical and the proof has already been presented in Proposition 1 of [1] but it is reminded here.

Proposition 1. *Let $\mathscr{H}^m = \mathbb{R}^n \times H^m \times H^{m-1}$ for $m > 0$.*
If there exists a norm on $\mathscr{H} = \mathscr{H}^1$ for which the linear operator associated to system (1.1) *is dissipative with $A + BK$ non singular, then there exists a unique solution (X, u, u_t) of system* (1.1) *with initial conditions $(X^0, u^0, v^0) \in \mathscr{H}^2$ assumed to be compatible with the boundary conditions. Moreover, the solution has the following regularity property: $(X, u, u_t) \in C(0, +\infty, \mathscr{H})$.*

The solutions belong to space $\mathscr{H}$ and consequently, one can introduce the following natural norm on this space:

$$\forall (X, u, v) \in \mathscr{H}, \quad \|(X, u, v)\|_{\mathscr{H}}^2 = |X|_n^2 + \|u\|^2 + c^2 \|u_x\|^2 + \|v\|^2,$$

where $|\cdot|_n$ is the classical euclidian norm of $\mathbb{R}^n$ and $\|\cdot\|$ refers to the L^2 norm. From this definition, the definition of exponential stability comes to be as follow:

Definition 1. A solution of system (1.1) belonging to $\mathscr{H}$ with the compatible initial conditions $(X^0, u^0, v^0) \in \mathscr{H}^2$ is exponentially stable if the following exponential estimate holds for $\gamma \geqslant 1, \delta > 0$ and $t > 0$:

$$\|(X(t), u(t), u_t(t))\|_{\mathscr{H}}^2 \leqslant \gamma e^{-\delta t} \|(X^0, u^0, v^0)\|_{\mathscr{H}}^2. \tag{1.4}$$

To prove the existence and uniqueness of the solution along with its exponential stability, the idea is to use a Lyapunov functional. This Lyapunov functional must be equivalent to $\|\cdot\|_{\mathscr{H}}$ and with a strictly negative derivative along the trajectories of the system. This last property ensures the dissipativity of system (1.1). These two properties are met if a LMI is satisfied. The following subsection gives modified versions of the originally LMI condition proposed in [1].

1.4.1 First result

An extension of Theorem 2 of [1] to reduce the conservatism is proposed as follows.

Theorem 1 *Consider system* (1.1) *with a given speed* $c > 0$, *a viscous damping* $c_0 > 0$ *and initial conditions* $(X^0, u^0, v^0) \in \mathscr{H}^2$ *compatible with the boundary conditions. Assume that, for a given integer* $N \in \mathbb{N}$, *there exist* $P_N \in \mathbb{S}_+^{n+2(N+1)}$ *and* $S, R \in \mathbb{S}_+^2$ *such that inequalities*

$$\begin{aligned}\Psi_N(A,B) = \mathrm{He}\left(Z_N^\top(A,B) P_N F_N\right) - c\tilde{R}_N \\ + c\left(H_N^\top(A,B)(S+R)H_N(A,B) - G_N^\top(A,B) S G_N(A,B)\right) \prec 0,\end{aligned} \quad (1.5)$$

$$\Phi_N = P_N + \mathrm{diag}(0_n, S, 3S, \dots, (2N+1)S) \succ 0, \quad (1.6)$$

hold, where

$$\begin{aligned}
F_N &= \begin{bmatrix} I_{n+2N+2} & 0_{n+2N+2,2} \end{bmatrix}, \qquad \mathscr{N}_N(A,B) = \begin{bmatrix} A+BK & \tilde{B} & 0_{n,2(N+1)} \end{bmatrix},\\
\tilde{R}_N &= \mathrm{diag}\left(0_n, R, 3R, \cdots, (2N+1)R, 0_2\right), \qquad \tilde{B} = \tfrac{1}{2c} B \begin{bmatrix} 1 & -1 \end{bmatrix},\\
Z_N(A,B) &= \begin{bmatrix} \mathscr{N}_N^\top(A,B) & c\mathscr{Z}_N^\top(A,B) \end{bmatrix}^\top,\\
\mathscr{Z}_N(A,B) &= \mathbb{1}_N H_N(A,B) - \bar{\mathbb{1}}_N G_N(A,B) - \begin{bmatrix} 0_{2N+2,n} & L_N & 0_{2N+2,2} \end{bmatrix},\\
L_N &= \begin{bmatrix} \ell_{0,0} I_2 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ \ell_{N,0} I_2 & \cdots & \ell_{N,N} I_2 \end{bmatrix}, \qquad \mathbb{1}_N = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}, \ \bar{\mathbb{1}}_N = \begin{bmatrix} I_2 \\ \vdots \\ (-1)^N I_2 \end{bmatrix},\\
G_N(A,B) &= \begin{bmatrix} 0_{2,n+2N+2} & g \end{bmatrix} + \begin{bmatrix} K \\ 0_{1,n} \end{bmatrix} \mathscr{N}_N(A,B), \qquad g = \begin{bmatrix} 0 & 1 \\ 1+cc_0 & 0 \end{bmatrix},\\
H_N(A,B) &= \begin{bmatrix} 0_{2,n+2N+2} & h \end{bmatrix} + \begin{bmatrix} 0_{1,n} \\ K \end{bmatrix} \mathscr{N}_N(A,B), \qquad h = \begin{bmatrix} 1-cc_0 & 0 \\ 0 & -1 \end{bmatrix}.
\end{aligned} \quad (1.7)$$

Then, there exists a unique solution to the coupled infinite dimensional system (1.1) *and it is exponentially stable in the sense of norm* $\|\cdot\|_{\mathscr{H}}$, *i.e. there exist* $\gamma > 1$ *and* $\delta > 0$ *such that energy estimate* (1.4) *holds.*

Remark 2. It is worth mentioning that this theorem introduces a hierarchy in the stability conditions, that means all the systems stable using the previous theorem at an order N_0 are also stable for all higher orders $N \geqslant N_0$.

This theorem comes from the following Lyapunov functional at a given order $N \geqslant 0$:

$$V_N(X_N, u) = X_N^\top P_N X_N + \int_0^1 \chi^\top(x)\,(S + xR)\,\chi(x) dx, \quad (1.8)$$

where $\chi(x) = \begin{bmatrix} u_t(x) + cu_x(x) \\ u_t(1-x) - cu_x(1-x) \end{bmatrix}$ is a modified Riemann invariant for system (1.1) and X_N is an extended state made up of the ODE state X and projections of the infinite dimension state χ on a basis of shifted Legendre polynomials on Ω. We denote this projections as follows:

$$\mathfrak{X}_k := \langle \chi, \mathscr{L}_k \rangle = \int_0^1 \chi(x)\mathscr{L}_k(x)dx,$$

where $\mathscr{L}_k$ is the k^{th} Legendre polynomials. Introducing an extended state, we get:

$$X_N = \begin{bmatrix} X^\top \ \mathfrak{X}_0^\top \ \cdots \ \mathfrak{X}_N^\top \end{bmatrix}.$$

Proof. This part is inspired by [13, 20] and aims at reducing the conservatism introduced by the positivity condition obtained in [1]. The desired property on the Lyapunov functional is as follow:

$$\begin{aligned} \varepsilon_1 \|(X,u,u_t)\|^2_{\mathscr{H}} \leqslant V_N(X_N,u) \leqslant & \quad \varepsilon_2 \|(X,u,u_t)\|^2_{\mathscr{H}}, \\ \dot{V}_N(X_N,u) \leqslant & -\varepsilon_3 \|(X,u,u_t)\|^2_{\mathscr{H}}, \end{aligned}$$

The existence of ε_2 and ε_3 strictly follows the same lines than in [1]. The difference between Theorem 2 of [1] and this one lies in the existence of ε_1 because of the relaxed condition $\Phi_N \succ 0$ compared to $P_N \succ 0$.

As $S, R \in \mathbb{S}^2_+$ and $\Phi_N \succ 0$, there exists $\varepsilon_1 > 0$ such that the following inequality holds :

$$\begin{aligned} &\Phi_N \succeq \varepsilon_1 \left(\operatorname{diag}\left(I_n, 0_{2(N+1)}\right) + \tfrac{2+c^2}{2c^2} \operatorname{diag}\left(0_n, I_{2(N+1)}\right) \right), \\ &\forall x \in [0,1], \quad S + xR \ \succeq \ S \ \succeq \ \varepsilon_1 \tfrac{2+c^2}{2c^2} I_2. \end{aligned}$$

Then, these inequalities and equation (1.8) leads to:

$$\begin{aligned} V_N(X_N,u) \geqslant X_N^\top \Phi_N X_N + \varepsilon_1 \left(|X|_n^2 + \frac{2+c^2}{2c^2} \|\chi\|^2 \right) \\ + \int_0^1 \chi^\top(x)(S-\delta)\chi(x)dx - \sum_{k=0}^N (2k+1)\mathfrak{X}_k^\top (S-\delta)\mathfrak{X}_k, \end{aligned}$$

with the shorthand notation $\delta = \varepsilon_1 \frac{2+c^2}{2c^2} I_2$. Using Lemma 1 together with the norm equality $\|\chi\|^2 = 2\left(\|u_t\|^2 + c^2\|u_x\|^2\right)$ implies:

$$V_N(X_N,u) \geqslant \varepsilon_1 \left(|X|_n^2 + \|u_t\|^2 + (2+c^2)\|u_x\|^2 \right).$$

Finally, using Lemma 1 of [1] brings the conclusion: $V_{N,\alpha}(X_N,u) \geqslant \varepsilon_1 \|(X,u,u_t)\|^2_{\mathscr{H}}$.

Remark 3. Theorem 1 is less restrictive than Theorem 2 of [1] because the condition $P_N \in \mathbb{S}_+^{n+2(N+1)}$ is included in $\Phi_N \succ 0$.

1.4.2 An equivalent formulation

The previous theorem is not affine in A and B and consequently, it cannot be easily extended into a robust formulation. Before stating the robust corollary, the following equivalent formulation is proposed.

Corollary 1. *Consider system* (1.1) *with a given speed $c > 0$, a viscous damping $c_0 > 0$, A and B given and known with initial conditions $(X^0, u^0, v^0) \in \mathscr{H}^2$ compatible with the boundary conditions.*
Assume that, for a given integer $N \in \mathbb{N}$, there exist $P_N \in \mathbb{R}^{n+2(N+1)\times n+2(N+1)}$, $S, R \in \mathbb{S}^2_+$, $Y_N \in \mathbb{R}^{2\times(n+2N+6)}$ and $\mu > 0$ such that the following LMIs are satisfied:

$$\begin{cases} \Phi_N = P_N + \mathrm{diag}(0_n, S, 3S, \ldots, (2N+1)S) \succ 0, \\ \Theta_N(A,B) \prec 0, \end{cases} \tag{1.9}$$

with

$$\begin{aligned} \Theta_N(A,B) &= \begin{bmatrix} \mathscr{M}_N(A,B) + \mathrm{He}(Y_N^\top \mathscr{W}_N(A,B)) & Y_N^\top & \mu \mathscr{W}_N^\top(A,B) \\ \star & -\mu I_2 - cS & 0_2 \\ \star & \star & -\mu I_2 \end{bmatrix}, \\ \mathscr{M}_N(A,B) &= \begin{bmatrix} \mathrm{He}\left(Z_N^\top(A,B) P_N F_N\right) - c\tilde{R}_N & H_N^\top(A,B)(S+R) \\ (S+R)H_N(A,B) & -\frac{1}{c}(S+R) \end{bmatrix}, \\ \mathscr{W}_N(A,B) &= \begin{bmatrix} G_N(A,B) & 0_2 \end{bmatrix}, \end{aligned}$$

where $\star$ denotes the symmetric.
Then, there exists a unique solution to the coupled infinite dimensional system (1.1) *and it is exponentially stable in the sense of norm $\|\cdot\|_{\mathscr{H}}$, i.e. there exist $\gamma > 1$ and $\delta > 0$ such that energy estimate* (1.4) *holds.*

Proof. A similar strategy than the one developed in [7] is used here. Starting from Theorem 1, system (1.1) is exponentially stable if $\Phi_N \succ 0$ and $\Psi_N(A,B) \prec 0$. Ψ_N is not affine in A and B and it is apparently not possible to conclude whether or not it is even convex with respect to A and B. But, applying Schur complement, an equivalent formulation for $\Psi_N(A,B) \prec 0$ is:

$$\begin{bmatrix} \mathrm{He}\left(Z_N^\top(A,B) P_N F_N\right) - c\tilde{R}_N & H_N^\top(A,B)(S+R) \\ (S+R)H_N(A,B) & -\frac{1}{c}(S+R) \end{bmatrix} - c\mathscr{W}_N^\top(A,B) S \mathscr{W}_N(A,B) \prec 0.$$

$\mathscr{M}_N(A,B)$ is affine in A and B but it remains to treat the quadratic term $\mathscr{W}_N(A,B)$ which is till not affine in A and B. Using Lemma 2, the previous statement is equivalent to the existence of $\mu > 0$ and $Y_N \in \mathbb{R}^{2\times(n+2N+6)}$ such that the following hold:

$$\mathscr{M}_N(A,B) + \mathrm{He}(Y_N^\top \mathscr{W}_N(A,B)) + Y_N^\top(\mu I_2 + cS)^{-1} Y_N + \mu \mathscr{W}_N^\top(A,B) \mathscr{W}_N(A,B) \prec 0.$$

The second condition to get the equivalence is $\mu I_2 + cS \succ 0$ which is always guaranteed as $\mu > 0$ and $S \succ 0$.
Using two Schur complements on the first equation, the following equivalence is obtained for a given pair (A,B):

$$\Psi_N(A,B) \prec 0 \quad \Leftrightarrow \quad \Theta_N(A,B) \prec 0. \tag{1.10}$$

Remark 4. Corollary 1 is equivalent to Theorem 1. The two main differences are first in terms of complexity because there are two new variables Y_N and μ and secondly because it is affine in A and B.

1.4.3 Robustness analysis

Using the new LMI conditions obtain previously, we can extend Theorem 1 to get the robust exponential stability of uncertain system (1.1). One of the benefits of using a Lyapunov functional approach relies indeed on the possibility of extending the previous analysis to robustness analysis. The uncertainties on matrices A and B are assumed to be of polytopic type, as described in equation (1.3). That leads to the following result.

Corollary 2. *Consider uncertain system* (1.1) *with a given speed $c > 0$, a viscous damping $c_0 > 0$ and initial conditions $(X^0, u^0, v^0) \in \mathscr{H}^2$ compatible with the boundary conditions. The uncertain couple $[A\ B]$ satisfies equation* (1.3)*.*
Assume that, for a given integer $N \in \mathbb{N}$, there exist $P_N \in \mathbb{R}^{n+2(N+1)\times n+2(N+1)}, S, R \in \mathbb{S}^2_+, Y_N \in \mathbb{R}^{2\times(n+2N+6)}$ and $\mu > 0$ such that the following LMIs are satisfied:

$$\begin{cases} \Phi_N = P_N + \mathrm{diag}(0_n, S, 3S, \dots, (2N+1)S) \succ 0, \\ \Theta_N(A_i, B_i) \prec 0 \text{ for all } i \in [1, m], \end{cases} \tag{1.11}$$

with the notation of Corollary 1.
Then, there exists a unique solution to the coupled uncertain infinite dimensional system (1.1) *and it is exponentially stable in the sense of norm $\|\cdot\|_{\mathscr{H}}$, i.e. there exist $\gamma > 1$ and $\delta > 0$ such that energy estimate* (1.4) *holds.*

Proof. The main advantage now is that Θ_N is affine in A and B if P_N, R, S, Y_N and μ are fixed. And consequently, it is convex with respect to A and B at a price of an increased complexity. As it is convex in A and B, if $\Theta_N(A_i,B_i) \prec 0$ for all $i = 1,\dots,m$, then $\Theta_N(A,B) \prec 0$. Indeed, if equation (1.3) holds, then we get:

$$\begin{aligned} &Z_N(A,B) = \textstyle\sum_{i=1}^m \lambda_i Z_N(A_i,B_i), \quad H_N(A,B) = \sum_{i=1}^m \lambda_i H_N(A_i,B_i), \\ &G_N(A,B) = \textstyle\sum_{i=1}^m \lambda_i G_N(A_i,B_i), \quad \text{and consequently,} \\ &\mathscr{M}_N(A,B) = \textstyle\sum_{i=1}^m \lambda_i \mathscr{M}_N(A_i,B_i),\ \mathscr{W}_N(A,B) = \sum_{i=1}^m \lambda_i \mathscr{W}_N(A_i,B_i). \end{aligned}$$

The previous considerations lead to $\Theta_N(A,B) = \sum_{i=1}^{m} \lambda_i \Theta_N(A_i,B_i)$. As for $i \in [1,m]$, $\Theta_N(A_i,B_i) \prec 0$ then $\Theta_N(A,B) \prec 0$. Corollary 1 can be applied, ensuring the exponential stability for all $[A\ B]$ in $\mathrm{Co}_{i=1,\dots m}\{[A_i\ B_i]\}$ and that ends the proof.

Remark 5. Conditions (1.11) must hold for all $i \in [1,m]$. That means $P_N, R, S, \mathscr{X}_N$ and μ are independent of i, making the condition highly conservative for large m or very different A and B matrices.

Corollary 2 is a robust corollary as A and B may vary inside an uncertainty set but the system keeps its exponential stability behavior. This result can also apply for some bounded time-varying A and B. The following section will give an example about the robustness property.

1.5 Examples

For the example section, we will study the following example taken from [13, 23]. This example is a regenerative chatter if the wave is a pure delay (for $cc_0 = 1$). The system is given as follows:

$$A(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-k & 10 & 0 & 0 \\ 5 & -15 & 0 & 0.25 \end{bmatrix}, B(k) = \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix} \text{ and } K = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\top}. \tag{1.12}$$

For $k = 1$ and different values of $c > 0$ and $c_0 > 0$, Corollary 1 gives Figure 1.2. The stability regions are in gray scale. The hatched areas are the exact unstable areas, they are obtained using the methodology described in [2]. As the order increases, more stability pockets are detected. Theorem 1 seems an effective tool for the stability analysis of coupled systems. The chart is different than in [1] because it gives a better illustration of the stability behavior as c ad c_0 vary. Indeed, it displays stability pockets which are known to be of difficult access for Lyapunov-like techniques [12]. Moreover, it is easier to compare with the results obtained with time-delay systems as τ is the delay and for $cc_0 = 1$, system (1.1) is a time-delay system (see [2] for more information).

For the robust stability analysis, we can use an interval for the k parameter. For instance, if $k \in (1,2)$, we get the following vertices: $[A(1)\ B(1)]$ and $[A(2)\ B(2)]$. We tested the two scenari $k \in (1,2)$ and $k \in (0.1,1.5)$ for some values of c and c_0. The results are displayed in Figure 1.3. The black area is stable for Corollary 2 with the desired uncertainties. The white area is unknown and the hashed area is an inner approximation of the unstable area. One can see that there exist areas for which the system is robustly stable to variation of k. To the best of our knowledge, there does not exist an exact robust stability criterion so we cannot compare with the exact

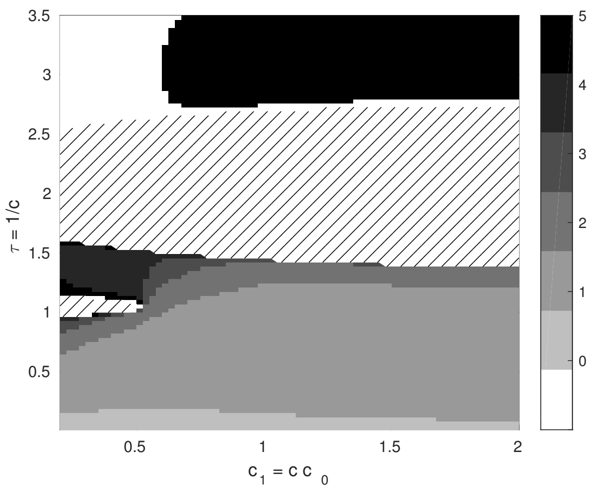


Fig. 1.2: Stability analysis using Theorem 1 with example (1.12) for $k = 1$. The hatched area is the exact unstable area. The scale of gray is the stable area using Corollary 1 for $N = 0$ to 5. The white area is the stable area, undetected by the corollary up to the order 5.

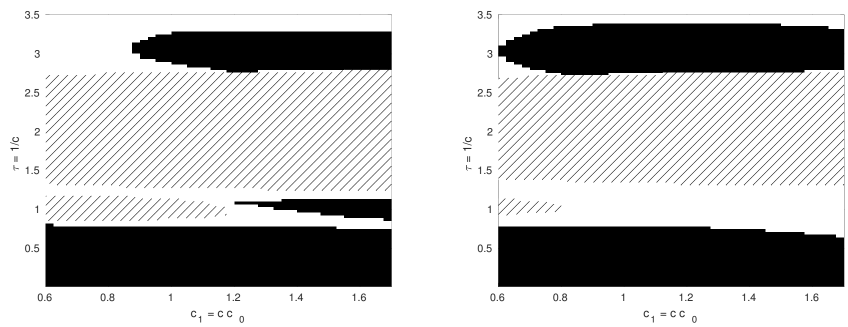
unstable area. This inner approximation is obtained as a summation of unstable area for some values of k.
Corollary 2 is able to detect stability pockets and seems to perform with a good accuracy because the stable area is very close to the hashed one. For $c_1 = cc_0 = 1$, an exact robust stability criteria can be achieved using the Control Toolbox of Matlab® together with the `allmargin` command as said in [20]. The results are displayed below and the stable intervals are written in terms of τ:

Scenario	$k \in (1,2)$	$k \in (0.1,1.5)$
`allmargin`	$(0,0.859) \cup (1.117,1.264) \cup (2.75,3.5)$	$(0,1.328) \cup (2.718,3.5)$
Cor. 2 ($N = 7$)	$(0,764) \cup (2.869,3.254)$	$(0,764) \cup (2.746,3.37)$

These two results are very close even if it misses one stability pocket for the first case. So we can say that polytopic uncertainties on A and B are well-addressed using Corollary 2.

1.6 Conclusion

As a conclusion, the present chapter addressed the problem of robust stability of an interconnected ODE/String system. The robust stability result is obtained consider-



(a) $k \in (1,2)$ (b) $k \in (0.1,1.5)$

Fig. 1.3: Stability areas for system (1.1) with matrices defined in (1.12) with $c \in (0.28,100)$ and $cc_0 \in (0.6,1.7)$. The black area is robustly stable in the considered interval while the white area is unknown at an order $N = 7$. The hashed area is an inner-approximation of the unstable area for the system. For (a), the hatched area corresponds to the exact unstable area for $k \in \{1,1.2,1.5,2\}$ obtained using a pole location argument. For (b), this areas is related to the unstable areas for $k \in \{0.1,0.5,0.8,1\}$.

ing polytopic uncertainties on the ODE part, i.e. on matrices A and B. The extended theorem proposes to enlarge the stability result by considering a relaxed positivity condition and the robust result takes advantage of the convex form of the LMIs. This corollary has been tested on a well-known example: a regenerative chatter. Comparing with exact stability results, the proposed methodology is indeed very accurate.

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