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New model transformations for the stability analysis of time-delay systems *

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Abstract: This paper deals with the stability analysis of time delay systems based on continuous-time approach. The originality of the present paper relies on the construction of several models for a same time-delay systems using the interconnection of an ordinary differential equation and a transport partial differential equation. The stability analysis is then performed using a Lyapunov functional. These models are constructed in order to first reduce potentially the complexity of the resulting stability conditions. Second several models are build in order to be interpreted as a discretization scheme as the one usually used in the Lyapunov functional. The proposed result can be seen as a generalized $(N - M)$ discretization which consists in both a time-discretization of the delay interval into $M$ sub-intervals, and the projection of the state function within each sub-interval on the Legendre polynomials of degree less than $N$. The efficiency of this novel approach is illustrated on an academic example.

Keywords: Time-delay systems, Model transformations, Transport equation, Lyapunov stability, Integral inequalities, Linear matrix inequalities.

1. INTRODUCTION

Time-delay systems have been widely investigated in many different areas, as in networked control systems, mechanical transmissions or biological systems. There is indeed a very large literature in time-delay system and we can refer to Xu et al. (2006), Mondie et al. (2005) and Gu et al. (2003) for instance, or the survey from Fridman (2014). Stability of time-delay systems has a crucial, practical and theoretical importance since the delay term can be a source of instability and poor performance of a system. Many stability studies and results have been proposed, as for instance in Chen (1995), Gao and Wang (2004) and Xu et al. (2001) (see also the references therein).

A relevant direction of research that was popular in the 2000's consists in applying several models transformation to a time-delay system in order to derive less conservative results as described for instance in Richard (2003). Among them the reader may look at Gu and Niculescu (2000) or at the descriptor representation introduced in Fridman and Shaked (2002). It is also well-known that these transformations may include some additional dynamics as clearly explained in Gu and Niculescu (2000).

Unlike the usual method based on the application of the Lyapunov-Krasovskii theorem, this paper aims at demonstrating the benefits of employing a model of time-delay systems, which is represented as the interconnection of an ordinary differential equation and a transport equation. This idea is obviously not new (see for instance Krstic (2009)). In fact, the transport equation allows to express the delay term of the time-delay system in a coupled system. More precisely, consider the following time delay system

\[ X(t) = \int_{0}^{t} A X(t) + A_d z(t-h) \, dt \]

where $X(t) \in \mathbb{R}^{n \times n}$ is the state of the system and the matrices $A, A_d \in \mathbb{R}^{n \times n}$ are constant and where $\phi$ represents the initial conditions of the delay system. It is easy to show that the delay term $X(t-h)$ of (1) could be expressed through a transport equation taking the inverse of the delay as the transport velocity. Let us indeed consider the following transport equation over the space domain $(0, 1)$:

\[ h \partial_z z(x,t) + \partial_t z(x,t) = 0, \quad x \in (0,1), t > 0, \]

which can be illustrated by Figure 1. The left boundary in $x = 0$ has the state $X(t)$ of the ODE input in $z(0,t)$ of the transport equation (2), and it takes time before reaching the output $z(1,t)$ at the right boundary in $x = 1$. This amount of time corresponds here to the delay $h$. Therefore, we obtain the delayed term $X(t-h)$ at the output $z(1,t)$ of the transport equation (2). Thus, system (1) is equivalent to the following coupled ODE-PDE system

\[
\begin{cases}
    \dot{X}(t) = AX(t) + A_d z(t-h), & t > 0, \\
    h \partial_t z(x,t) + \partial_x z(x,t) = 0, & x \in (0,1), t > 0, \\
    z(0,t) = X(t), & t > 0,
\end{cases}
\]

Control and stability of this specific kind of systems have been studied in many recent papers in literature in differ-

Fig. 1. Illustration of system (3) equivalent to time-delay system (1).

\[ \dot{X}(t) = AX(t) + A_d z(t-h), \quad t \geq 0 \]

\[ X(\theta) = \phi(\theta), \quad \theta \in [-h, 0] \]
ent applicative fields. For example, in hydraulic domain, the paper Coclite et al. (2005) treats a fluidodynamic model for traffic flow using wave front tracking approach and Gugat et al. (2011) designs a feedback laws stabilising a fan-shaped network given by a coupled ODE-PDE system. The works presented in Baudouin et al. (2016) and Safi et al. (2017b) give a hierarchy stability conditions of the coupled system above using linear matrix inequalities depending on different parameters of the coupled system.

The purpose of this paper consists in exploiting the stability result of Safi et al. (2017b) and applying a discretization scheme on the delayed term of system (1). In other words, we will divide the delay interval \([-h,0]\) of system (1) in \(M\) sub-intervals \([-h, \frac{-M}{M-1}h], \ldots, [-\frac{2h}{M}, -\frac{h}{M}], [-\frac{h}{M}, 0]\), and we will project the delayed state corresponding to each sub-interval on the first \(N + 1\) Legendre polynomials. Applying a \(M\)-discretization of the delay interval and \(N\)-sized projection on Legendre polynomials will allow to compare the efficiency of each method and could lead at evaluate an optimal value pair \((M, N)\) giving the best results for a given compromise between calculability and efficiency.

The document is presented as follows. The next section introduces briefly the tools used in this work. Section 3 formulates the problem, provides some preliminaries and exposes the main results of the paper Safi et al. (2017b) on the stability analysis of (1) using the coupled system approach. Section 4 itemizes the different models of the coupled ODE-PDE system allowing to express the time-delay system (1). Afterwards, Section 5. These results are applied on academic example in section 6. Finally, the last section draws some conclusions and perspectives.

**Notations:** \(\mathbb{N}\) is the set of positive integer, \(\mathbb{R}^n\) is the \(n\)-dimensional Euclidean space with vector norm \(||\cdot||\), \(I_n\) is the identity matrix in \(\mathbb{R}^{n \times n}\), \(0_{m,n}\) the null matrix in \(\mathbb{R}^{m \times n}\), \(A^T\) replaces the symmetric matrix \([B^\dagger B]\). We denote \(\mathbb{S}^n \subset \mathbb{R}^{n \times n}\) (resp. \(\mathbb{S}^m_+ = \{P \in \mathbb{S}^n, P > 0\}\)), and \(\mathbb{D}_+^m\) the set of symmetric (resp. symmetric positive definite and diagonal positive definite) matrices and \(\mathrm{diag}(A, B)\) is a bloc diagonal matrix. For any square matrix \(A\), we define \(\mathrm{He}(A) = A + A^T\). Finally, \(L^2(0, 1; \mathbb{R}^m)\) represents the space of square integrable functions over the interval \([0, 1]\) with values in \(\mathbb{R}^m\) and the partial derivative in time and space are denoted \(\partial_t\) and \(\partial_r\), while the classical derivative are \(X = \frac{d}{dt}X\) and \(L' = \frac{d}{dx}L\). We first define the following set of matrices commuting with a matrix \(A \in \mathbb{D}_+^m\) as follows \(\mathcal{M}_A^m := \{M \in \mathbb{S}^m_+, M A = A M\}\). The notation \(X_t\) stands for \(X_t(\theta) = X(t+\theta)\) for all \(\theta \in [-h, 0]\) where \(h\) is a positive scalar.

### 2. PRELIMINARIES

#### 2.1 Legendre polynomials

Using a formulation of time-delay system (1) as a coupled ODE-transport PDE system like (3), one will need to work in the functional space \(L^2(0, 1)\). To conduct this stability study, we choose the shifted Legendre polynomials since they form an orthogonal base of \(L^2(0, 1)\) with the canonical scalar product

\[
< \mathcal{L}_j, \mathcal{L}_k > = \int_0^1 \mathcal{L}_j(x) \mathcal{L}_k(x) \, dx = \frac{1}{2k+1} \delta_{jk},
\]

where \(\delta_{jk}\) is the Kronecker’s coefficient, equal to 1 if \(j = k\) and 0 otherwise. Legendre polynomials are denoted \(\{\mathcal{L}_k\}_{k \in \mathbb{N}}\) and they are characterised by the boundary values \(\mathcal{L}_0(0) = (-1)^k, \mathcal{L}_1(1) = 1\) and their differentiation for all \(k \geq 1\)

\[
\mathcal{L}'_k(x) = \sum_{j=0}^{k-1} \ell_{kj} \mathcal{L}_j(x)
\]

\[
\ell_{kj} = \begin{cases} 
(2j+1)(1-(-1)^{k+j}), & \text{if } j \leq k-1, \\
0, & \text{if } j \geq k.
\end{cases}
\]

The shifted Legendre polynomials will provide basis of polynomials that will be used to project the state \(z\) of the transport equation for the purpose of the stability study.

#### 2.2 Bessel-Legendre inequality

The Bessel inequality based on Legendre polynomials is characterised by the positivity of the error between the complete norm of an \(L^2(0, 1)\) vector and its projection on Legendre polynomials. The result is given by

**Lemma 1.** Let \(u \in L^2(0, 1; \mathbb{R}^m)\) and \(R \in \mathbb{S}^m_+\). Then the following inequality holds for all \(N \in \mathbb{N}\)

\[
\int_0^1 u^T(x) R u(x) \, dx \geq U_N^T \begin{bmatrix} R & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (2N+1)R \end{bmatrix} U_N, \tag{4}
\]

where

\[
U_N = \begin{bmatrix} f_0^1 u(x) \mathcal{L}_0(x) \, dx \\ \vdots \\ f_0^1 u(x) \mathcal{L}_N(x) \, dx \end{bmatrix} \in \mathbb{R}^{m(N+1)}.
\]

In our context, this basic lemma will allow to prove that a computable stability criteria based on linear matrix inequalities (LMIs) proves actually the exponential stability of the whole ODE-PDE system under consideration.

#### 2.3 Definition : exponential stability of system (3)

Considering the finite dimensional system in \(X(t)\) coupled to the transport equation in the variable \(z(x,t)\), the expression of \(E(t)\) the total energy of this coupled ODE-transport PDE system is \(E(t) = ||X(t)||_{L^2}^2 + ||z(x)||_{L^2(0,1;\mathbb{R}^m)}^2\). System (3) is exponentially stable if there exist constant \(K > 1\) and \(\delta > 0\), depending on different parameters of the system, such that \(E(t) \leq Ke^{-\delta t}E(0)\). Thus, the total energy \(E(t)\) of system (3) admits a decreasing upper bound and the system’s state converges exponentially. These tools and their proofs are more detailed in Safi et al. (2017b).

### 3. PRELIMINARY ON STABILITY OF COUPLED TRANSPORT ODE-PDE SYSTEMS

Consider now the following more general coupled ODE-transport PDE system

\[
\begin{cases}
\dot{X}(t) = AX(t) + Bz(1,t) & t > 0, \\
\partial_x z(x,t) + A_2 z(x,t) = 0, & x \in (0,1), t > 0, \\
z(0,t) = CX(t) + Dz(1,t), & t > 0, \\
z(x,0) = z_0(x), & x \in (0,1), \\
X(0) = X^0.
\end{cases}
\]

where the state is composed by the finite variable \(X(t) \in \mathbb{R}^n\) of the ODE and the infinite dimensional state \(z(\cdot, t) \in L^2(0, 1; \mathbb{R}^m)\) of the transport PDE. The matrices \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}\) and \(D \in \mathbb{R}^{m \times m}\) are
constant. The transport matrix $\Lambda \in \mathbb{D}_+^n$ of the PDE is constant and diagonal. In Safi et al. (2017b), sufficient conditions were provided to guarantee the $L^2$ stability of system (5). This result uses Lyapunov method based on Bessel-Legendre inequality and Legendre polynomials ($L_k$) to approximate the infinite dimensional state $z(x,t)$ of the PDE. This method provides LMIs guaranteeing stability of the coupled system led by the choice of an appropriate Lyapunov functional. This Lyapunov functional is indexes by the order of approximation $N$ and is given by

$$V_N(t) = V_{N,1}(t) + V_{N,2}(t),$$

(6)

where

$$V_{N,1}(t) = \begin{bmatrix} \mathcal{Z}_N(t) \end{bmatrix}^T \begin{bmatrix} P & Q_N \\ T_N & 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}(t) \\ \mathcal{Z}_N(t) \end{bmatrix}$$

and

$$V_{N,2}(t) = \int_0^t z^T(x,t)e^{-2\delta s\Lambda^{-1}}(S + (1 - x)R)z(x,t)dx.$$

First of all, the proof is based on the proposed Lyapunov functional (6) and the use of Lemma 1. Moreover, the objective is to prove that the Lyapunov functional $V_N$ is an equivalent norm of the total energy $E_X(t, z(t))$ and decreases exponentially to zero. This is formalized by the following inequalities

$$\delta_1 E(t) \leq V_N(t) \leq \delta_2 E(t),$$

$$V_N(t) + 2\delta_2 V_N(t) \leq -\delta_3 E(t),$$

(11)

for some positive scalars $\delta_1$, $\delta_2$, which are guaranteed by $\Phi_N(\Lambda, \delta) > 0$ and $\delta_3$, by the $\psi_N(\Lambda, \delta) < 0$. $\square$

Remark 1. The stability condition of Theorem 1 are valid for any integer $N$; As in Seuret and Gouaisbaut (2015), it was also proven in Safi et al. (2017b) that these conditions form a hierarchy. In other words, increasing $N$ can only reduce the conservatism of the condition.

Remark 2. It is worth to note that there exists a relation between the delayed version of ODE state $X$ and the functional state $z$. Indeed solving the partial differential equation, one easily obtains that $z(x,t) = X(t - h(x))$. This also means that the Lyapunov functional presented in (6) can be easily translated as a usual Lyapunov-Krasovskii functionals.

3.1 $L^2$-norm stability of system (5)

We note that Theorem 1 ensures the $L^2$ stability of the ODE-PDE system. This stability is weaker the Lyapunov-Krasovskii stability, which relies on the supremum norm. However, $L^2$ stability for system (5) implies stability in the sense of Lyapunov-Krasovskii Theorem. This can be done by recalling that $X(t - \theta) = z(t, \theta/h)$ and by noting that

$$|X(t)|^2 > E(t) \leq \eta \sup_{s \in [-h,0]} |X(s)|^2,$$

and the inequalities (11) also imply

$$\delta_1 |X(t)|^2 \leq V_N(X, z) \leq \delta_2 \eta \sup_{s \in [-h,0]} |X(s)|^2,$$

$$V_N(X, z) + 2\delta_2 V_N(X, z) \leq -\delta_3 |X(t)|^2.$$

This shows that this formulation does not bring any restriction with the usual Lyapunov-Krasovskii Theorem.

As presented in the introduction, this paper aims at presented several ODE-PDE models that are equivalent to the original time-delay system (1). Then, based on the stability conditions of Theorem 1, a simple procedure to design discretized Lyapunov-Krasovskii functionals mixed with the framework of the Bessel-Legendre inequality.

4. MODEL TRANSFORMATIONS

Let us first recall the model provided in the introduction. It was mentioned there that the system (1) writes as the ODE-PDE system (5) with the following matrices, besides $A$ which is unchanged

$$B = A_n, \quad C = I_n, \quad D = 0_n.$$

This is obviously not the only way to model (1) as (5). Some relevant ones are presented below.

4.1 First modelling

In the situation where matrix $A_d$ is not full rank, it is possible to reduce the complexity of the resulting stability conditions by noting that matrix $A_d$ can be rewritten as

$$A_d = \begin{bmatrix} A_d \end{bmatrix},$$

(9)

then, system (5) is exponentially stable. Moreover, for a given transport speed matrix $\Lambda \in \mathbb{D}_+^n$, there exists a constant $K > 0$ and a guaranteed decay rate $\delta > \delta$ such that the energy of the system verifies, $\forall t > 0$,

$$E(t) \leq K e^{-2\delta t} \left( |z(0)|^2 \right. \left. + \|z(0)\|^2 \right).$$

(10)

Proof : The proof of this stability result is fully detailed in Safi et al. (2017b) and is not recalled in the present paper. Nevertheless a sketch of the proof is included to latter illustrate the relationship between this result and the Lyapunov-Krasovskii stability.
In this model, illustrated in Figure 2(b), the matrices $A$ and $C$ are two matrices in $\mathbb{R}^{n \times m}$ and $\mathbb{R}^{m \times n}$ respectively, where $m$ is the rank of $A_d$. Hence, in this situation, the ODE-PDE model (5) with

$$B = B_d, \quad C = C_d, \quad D = 0_m.$$ 

and again with the same matrix $A$, is still able to represent the time-delay system (1). This modification only implies a light modification in the model as depicted in Figure 2(a). Therefore, applying the stability conditions of Theorem 1 on this particular model can be interpreted as a mixture of the Bessel-Legendre inequality and a discretization process of the delay interval. Indeed, one may look at the structure of the associated Lyapunov functional leading to this stability conditions.

Again, the computation of the solution of the transport PDE in (5) with $\Lambda = \frac{2}{h} I_{2m}$ yields

$$z(x, t) = \begin{bmatrix} C_d X(t - \frac{h}{2} x) \\ C_d X(t - \frac{h}{2} - \frac{h}{2} x) \end{bmatrix}. \tag{12}$$

Therefore, re-injecting the expression of $z$ in the definition of the Lyapunov functional leads to

$$V_N(X_N(t), z(t)) = X(t) \gamma(t) \begin{bmatrix} P Q_N \end{bmatrix} \begin{bmatrix} X(t) \\ X_N(t) \end{bmatrix} + \int_{t-h}^{t} \int_{t-h}^{t} e^{-\delta \tau} \delta x (S + (1-x)R) \begin{bmatrix} X(t-h \tau x) \\ X(t - \frac{h}{2} x) \end{bmatrix} dx.$$ 

where the augmented state $X_N$ corresponds to the projection of the state $z$ given in (12) on the Legendre polynomials of degree less than $N$. More especially, an expression of this vector is given, after a change of variable, by

$$X_N = \begin{bmatrix} \int_{t-h}^{t} C_d X(t-h\tau x) L_0(x) dx \\ \vdots \\ \int_{t-h}^{t} C_d X(t-h\tau x) L_N(x) dx \end{bmatrix}.$$ 

Applying the changes of variable $s = t - \frac{h}{2} x$ and $s = t - \frac{h}{2} - \frac{h}{2} x$ to the previous expression, the augmented vector $X_N$ can be rewritten as follows

$$X_N(t) = \begin{bmatrix} \int_{t-h}^{t-\frac{h}{2}} C_d X(s)L_0(\frac{2\tau s}{h}) ds \\ \int_{t-h}^{t} C_d X(s)L_0(\frac{2\tau s}{h}) ds \end{bmatrix}.$$ 

and apply the change of variable $s = t - \frac{h}{2} x$ to the last integral term of $V_N$, the following expression of the functional is obtained and is consistent with the definition of a Lyapunov-Krasovskii functional

$$V_N(X(t)) = \begin{bmatrix} X(t) \\ X_N(t) \end{bmatrix} \begin{bmatrix} P Q_N \end{bmatrix} \begin{bmatrix} X(t) \\ X_N(t) \end{bmatrix} + \frac{2}{h} \int_{t-h}^{t} \int_{t-h}^{t} e^{-\delta \tau} (S + \frac{h}{2} (s-t+\frac{h}{2}) R) \begin{bmatrix} X(s) \\ X(s-\frac{h}{2}) \end{bmatrix} ds,$$

These comments demonstrate the potential of the ODE-PDE modeling of time-delay delay systems. Indeed, it has been shown that the simple and understandable modifications have a strong impact on the interpretation of the stability conditions developed in Theorem (1). A first aspect deals with the reduction of complexity of the LMI (due to both a reduction of the size and of the number of
decision variables) if the delay matrix $A_d$ is not full rank. Second, it has been showed that if is possible to construct an augmented model which must verify the constraint $B D C = A_d$ in order to provide an equivalent formulation of the time-delay system. It can also be understood that this decomposition $(A,B,C,D,L)$ is not unique and many other models can be generated. In the next section, a general formulation of the discretization is provided to extend the discretization process to any order $M \in \mathbb{N}$.

4.3 General discretization process

Here, we define a new parameter $M$, which corresponds to the number of subintervals of $[0,1]$ to be considered. Following the previous section, let us consider the following model transformation, which allows to express the time-delay system (1) as in the ODE-PDE system (5) with the following matrices.

$$
B = \begin{bmatrix} 0_{n,(M-1)m} & B_d \end{bmatrix}, \\
D = \begin{bmatrix} 0_{m,(M-1)m} & 0_{m,(M-1)m} \\
I_{(M-1)m} & 0_{(M-1)m,m} \end{bmatrix}, \\
C = \begin{bmatrix} C_d \\
0_{(M-1)m,n} \end{bmatrix}, \\
\Lambda = \frac{M}{N} I_{Mm}.
$$

Again, it can be easily seen that $B D C = B_d C_d = A_d$. A graphical interpretation of this model is given in Figure 2(c). It can be seen there that the boundary conditions impose the following constraints, that expend the one provided in the previous model

$$
\begin{bmatrix}
z_1(0,t) \\
z_2(0,t) \\
\vdots \\
z_M(0,t)
\end{bmatrix} = \begin{bmatrix} C_d X(t) \\
z_1(1,t) \\
\vdots \\
z_{M-1}(1,t)
\end{bmatrix}
$$

Following the same arguments as in the previous section, we get that

$$\forall i = 1, \ldots, M-1, \quad z_{i+1}(0,t) = z_i(1,t) = z_i(0,t - \frac{h}{M}),$$

where we used the solutions of the transport equation. Hence, re-injecting these expressions and by injection, we get that

$$\forall i = 1, \ldots, M, \quad z_i(0,t) = C_d X(t - \frac{(i-1)h}{M})$$

and $z_M(1,t) = C_d X(t - h)$.

Hence, this model transformation on the ODE-PDE system allows to include some intermediate values of the state function $X_t$. It is also worth noting that the associated Lyapunov functions can also be rewritten as a Lyapunov-Krasovskii functionals as presented in the previous section. The computation leading to this functional follows exactly the same procedure and is not therefore not presented here.

A last comment deals with the complexity of the resulting stability conditions. Increasing both $N$ and $M$ notably increase the size of the LMI as well as the number of decision variables. The number of decision variables $DV$ involved in the discretization with respect to any integer $M$ at any order $N$ of the Legendre polynomials is given by

$$DV = \frac{1}{2} \left( n^2 + n + Mm(3N + 5) + (Mm)^2 ((N+1)^2 + 2) \right) \quad (14)$$

5. NUMERICAL EXAMPLES

Consider the time-delay system (1) with the following matrices

$$A = \begin{bmatrix} -2 & 0 \\
0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\
-1 & -1 \end{bmatrix}.$$


