Event-triggered PI control for continuous plants with input saturation

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***Abstract*— This paper proposes a methodology to design stabilizing event-trigger strategies for PI controlled linear continuous-time plants subject to input saturation. Using Lyapunov theory techniques, LMI-based conditions are derived to guarantee regional (or global, when possible) asymptotic stability of the origin. These conditions can be cast in an optimization problem to choose the parameters of the trigger function aiming at reducing the sampling activity, while ensuring the regional stability of the origin with respect to a given set of admissible initial states. Simulation results illustrate the application and potentialities of the method.**

I. Introduction

Event-triggered control techniques consist in sampling and transmitting data only when a trigger condition occurs [1]. A challenge in this case is to devise trigger conditions that ensure the stability of the closed-loop system under the aperiodic sampling strategy. With this respect, [2] is a widely cited work, which shows that there exist lower bounds for the inter-sampling time of a stabilizing event-triggered control when a threshold on the relative state measurement error is used as trigger condition. Moreover, it is of major interest in networked control systems to develop systematic methods to "tune" the parameters of the trigger condition aiming at reducing the number of samples to deal with the problems of limited bandwidth and energy consumption (mainly in wireless networks). More recently, other issues have also been addressed in the literature. For instance, the co-design between the trigger function and controller gains [3] and the application to nonlinear systems along with the assumption that there is one sensor and one event-trigger generator for each state variable [4].

On the other hand, PI controllers are of special interest due to its massive use in the process industry and applications where constant references and disturbances have to be tracked or rejected. Among the works dealing with event-triggered PI or PID controlled systems, we can cite [5] and [6] that propose triggering data transmissions when the difference between the current plant output value and the last sampled one crosses a given threshold. Although simulations that give insights on the applicability of this strategy scheme are presented, those papers do not give explicit guidance on how to choose the threshold values neither prove the stability of the resulting closed-loop systems. In [7], the authors present a design method for event-triggered PI controllers and LTI plants. Based on the Lyapunov Theory, formal proofs of stability of the closed-loop system under the proposed sampling strategy are provided. The proposed method allows to design all the parameters of the controller and an event-trigger condition to attain a linear quadratic performance criterion. However, the trigger function used is somewhat complex to calculate, as it uses a matrix of dimension $3(n+1)$. In [8] and [9], stability analysis of event-triggered PI controllers acting on first-order stable plants are presented; the measurement error is used in the trigger criterion without normalizing it with respect to the state. In this case, tracking of constant references with zero steady state error is not achieved.

Furthermore, control input saturation is an ubiquitous phenomenon in real control systems. Thus, it is natural to investigate the use of event-triggered techniques with plants subject to control input saturation. In this case, the closed-loop system becomes nonlinear and for exponentially unstable open-loop systems only regional stabilization is possible to achieve [10]. In this context, considering state-feedback control laws, in [11] algorithms based on linear-quadratic criteria are proposed to design periodic, event-triggered and self-triggered controllers. Lyapunov and hybrid system techniques are used along with generalized sector conditions to ensure exponential stability of the origin and to provide estimates of the region of attraction. Considering a discrete-time framework, saturating state feedback control laws are also addressed in [12]. Regarding PI controllers we can cite [13], which proposes to design an event-triggered controller without taking the saturation into account and then investigates how the saturation impacts the performance of the closed-loop system. The authors show, by means of examples, that the use of anti-windup techniques can reduce the negative effects of saturation. Since the saturation is not taken into account in the design of the trigger function, there is no *a priori* guarantee that the designed system will still be stable when the saturation effectively occurs. Moreover, no systematic procedure to determine suitable values for the trigger function are presented. In [14] and [15], stability issues under control saturation are addressed considering PI and generic dynamic output feedback controllers. Although the proposed conditions ensure that the trajectories are bounded in an ellipsoidal set, the convergence to the origin (i.e. asymptotic stability) is not guaranteed. It should be pointed out that the problem setup in that paper assumes

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that the output of the controller is continuously applied to the plant (i. e. no sampling and zero order hold are considered), which presupposes that controller and plant are in the same node in a networked implementation.

The present paper addresses the design of an event-trigger strategy for PI controlled continuous-time linear plants subject to input saturation. Differently from [14] and [15], we suppose that plant and controller are in different nodes and that the value of the control signal is kept constant between two sampling instants (i. e. between two successive events). The considered trigger function takes into account a weighted (through the use of generic positive definite matrices) relative distance between the last sampled state and the current continuous one. Based on Lyapunov theory, LMI conditions to ensure the asymptotic stability of the closed-loop system (under the event trigger strategy) for a given set of admissible initial states are proposed. Since the weighting matrices of the trigger function appear explicitly in the LMIs, a convex optimization problem aiming at selecting them in order to reduce the sampling activity (i. e. the number of trigger events) is proposed. A numerical example illustrates the application of the method.

Notation: $\mathbb{R}$ represents the set of real numbers. For $v \in \mathbb{R}$, $\mathrm{sat}(v)$ is the classical symmetric saturation function with limits $\pm u_0$, with $u_0 > 0 \in \mathbb{R}$. A deadzone non-linearity is defined as $\phi(v) \triangleq \mathrm{sat}(v) - v$; thereof, we can state that $\mathrm{sat}(v) = v + \phi(v)$. A' denotes the transpose of matrix A. $\mathrm{He}\{A\} \triangleq A + A'$. The symbol $*$ stands for symmetric blocks within a matrix. $\mathrm{tr}(A)$ denotes the trace of matrix A. $\mathrm{diag}(X, Y)$ denotes the block-diagonal matrix composed by the blocks X and Y. $\mathcal{E}(P) = \{\xi \in \mathbb{R}^q;\ \xi' P \xi \leq 1\}$ denotes an ellipsoid defined from a matrix $P = P' > 0 \in \mathbb{R}^{q \times q}$.

II. Problem Statement

Consider a continuous-time single-input single-output linear plant defined by the following equations:

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t) \\ y(t) = C_p x_p(t) \end{cases} \tag{1}$$

where $x_p(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}$ is the input, limited in amplitude such that $-u_0 \leq u(t) \leq u_0$, with $u_0 > 0$; $y(t) \in \mathbb{R}$ is the controlled output; A_p, B_p and C_p are real-valued constant matrices of appropriate dimensions. We assume that the plant is observable and controllable.

The controller is a continuous-time PI controller, defined by the following state space representation:

$$\begin{cases} \dot{x}_c(t) = -y(t) \\ \nu(t) = k_i x_c(t) - k_p y(t) \end{cases} \tag{2}$$

where $x_c(t) \in \mathbb{R}$ is the controller state; $\nu(t) \in \mathbb{R}$ is the controller output and $k_p, k_i \in \mathbb{R}$ are the proportional and integral gains, respectively.

We consider a networked control implementation where plant and controller are in separate nodes and are connected through a general purpose network forming the closed-loop system depicted in Figure 1.

At instants determined by an event-trigger generator, a sample of the plant output is sent to the controller node.

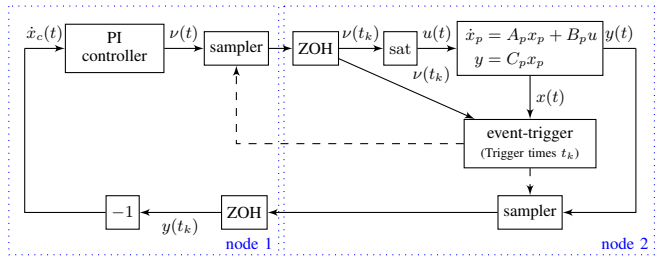

Fig. 1. Closed-loop system diagram

At the same time, the current value of $\nu(t)$ is sent to the plant input and to the event generator. Between two trigger instants, the controller and plant inputs are held at the constant sampled values by means of zero-order holders.

Therefore, the closed-loop system can be represented as follows:

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t) & \\ y(t) = C_p x_p(t) & \\ \dot{x}_c(t) = -y(t_k) & \forall t \in [t_k, t_{k+1}) \\ \nu(t) = k_i x_c(t) - k_p y(t_k) & \forall t \in [t_k, t_{k+1}) \\ u(t) = \mathrm{sat}(\nu(t_k)) & \forall t \in [t_k, t_{k+1}) \end{cases} \tag{3}$$

where t_k, $k = 0, 1, 2, 3, \ldots$ are the triggering times. We consider $t_0 = 0$. Note that the equilibrium point is assumed to be the origin. If the system is subject to a non-zero constant reference, the equilibrium is not the origin anymore, but a variable translation can be done and the stability analysis will be the same (see details in [7]).

Due to the control input saturation, the overall behavior of the closed-loop system is nonlinear. In this case, defining the state vector $x = [x_p' \ \ x_c']'$, the region of attraction for (3) can be defined as follows [16]:

Definition 1: The region of attraction (RA) of the origin for the system (3) is the set of all initial states $x_0 \in \mathbb{R}^{(n+1)}$ for which $x(0) = x_0 \implies x(t) \to 0$ as $t \to \infty$.

In words, the RA is the set of all initial conditions whose trajectories converge to the origin. Nevertheless the exact characterization of the RA is, in general, a complex task ([16]–[18]). Thus, it is useful to characterize subsets of the RA that have an analytical representation, such as ellipsoidal and polyhedral sets. These subsets can be used as estimates of the RA and are called regions of asymptotic stability (RAS) [16].

Considering system (3), we aim at designing an event-triggered control strategy (i. e. that defines the trigger instants t_k) to reduce the number of messages exchanged between the nodes, so that we can save energy and communication bandwidth, while keeping the closed-loop system stable. Hence we can formally state the problem we want to address as follows:

Problem 1: Devise an event-triggered strategy for the closed-loop system (3) guaranteeing the regional asymptotic stability of the origin for initial conditions in a given subset $\mathcal{X}_0$ of the RA of the closed-loop system, while reducing the number of data transmission events between the sensor/plant node and the controller node.

III. Continuous-time system stability

Before analyzing the event-triggered control strategy, we recall some stability conditions for the continuous-time system composed by the direct connection between (1) and (2), through a saturation function, i. e. $u(t) = \mathrm{sat}(\nu(t))$, which leads to the following system:

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p \,\mathrm{sat}(k_i x_c(t) - k_p C_p x_p(t)) \\ \dot{x}_c(t) = -C_p x_p(t) \end{cases} \tag{4}$$

We assume that the gains k_i and k_p have been designed such that (4) is regionally stable in a set $\mathcal{X}_0$. This is guaranteed if k_i and k_p are such that there exist a scalar $\varsigma > 0$, matrices $\bar{W} = \bar{W}' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$, $\bar{Z} \in \mathbb{R}^{1\times(n+1)}$ satisfying the following linear matrix inequalities (see [16] for details):

$$\begin{bmatrix} \mathrm{He}\{(A+BK)\bar{W}\} & \varsigma B - \bar{Z}' \\ * & -2\varsigma \end{bmatrix} < 0 \qquad \begin{bmatrix} \bar{W} & \bar{W}K' - \bar{Z} \\ * & u_0^2 \end{bmatrix} \geq 0 \tag{5}$$

and $\mathcal{X}_0 \subset \mathcal{E}(\bar{W}^{-1})$, with

$$A \triangleq \begin{bmatrix} A_p & 0 \\ -C_p & 0 \end{bmatrix};\ B \triangleq \begin{bmatrix} B_p \\ 0 \end{bmatrix};\ K \triangleq \begin{bmatrix} -k_p C_p & k_i \end{bmatrix} \tag{6}$$

In this case, it follows that $V(x) = x'Px$, with $P = \bar{W}^{-1}$ is such that $\dot{V}(x) < 0, \forall x \in \mathcal{E}(P)$, i. e. $\mathcal{E}(P)$ is a contractive domain for the closed-loop system (4).

IV. Event-trigger strategy

In this section, we propose an event-trigger strategy and provide stability conditions by means of a quadratic Lyapunov function.

Defining $\delta(t) \triangleq x(t_k) - x(t)$, with $x(t)$ being the state of the closed-loop system (in our case, the plant and controller system combined), it is shown in [2] that one can use the triggering criterion presented below to ensure that an event-triggered control system like (3) is stable:

if $||\delta(t)|| = \sigma_0 ||x(t)||$ **then**
  trigger
end if

where $||\cdot||$ denotes the Euclidean norm and σ_0 is a scalar selected from an interval of values determined by the dynamics of the system.

In this paper, we extend this idea by using the following generalized criterion:

if $\delta'(t)Q_\delta\delta(t) > x'(t)Q_x x(t)$ **then**
  trigger
end if

where Q_δ and Q_x are symmetric positive definite matrices of dimension $(n+1)\times(n+1)$. With this generalization, we add degrees of freedom and we expect to allow a larger reduction in the sampling activity. Notice that we can rewrite our event-trigger strategy as follows:

Algorithm 1 Event-trigger strategy

if $f(\delta, x) \triangleq \delta'(t)Q_\delta\delta(t) - x'(t)Q_x x(t) > 0$ **then**
  trigger
end if

The term $\delta'(t)Q_\delta\delta(t) - x'(t)Q_x x(t)$ is a relative measure of the deviation between the last sampled state and the current state; Q_δ and Q_x act as weights on this measure. The relation between these matrices plays a role similar to the one of σ_0 in [2], in the sense that the "larger" Q_x and the "smaller" Q_δ are, the more we let the current state deviate from the last sampled one and the less sampling activity is expected. To illustrate this, notice that if we choose $Q_x = \sigma I$ and $Q_\delta = \mu I$ we basically retrieve the criterion in [2] with $\sigma_0 = \sigma/\mu$ and, as shown in that paper, the larger σ_0 the less sampling activity is expected.

Let A, B and K be given matrices defined as in (6), with K verifying (5), and define

$$A_\delta \triangleq \begin{bmatrix} 0 & 0 \\ -C_p & 0 \end{bmatrix}$$

Now we derive conditions that ensure the asymptotic stability of the origin when the event-trigger strategy described in Algorithm 1 is used.

Theorem 1: If there exist a scalar $\varsigma > 0$, matrices $W = W' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$, $\bar{Q}_x = \bar{Q}_x' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$, $Q_\delta = Q_\delta' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$ and $Z, G_2 \in \mathbb{R}^{1\times(n+1)}$ such that the following LMIs are satisfied:

$$\begin{bmatrix} \mathrm{He}\{(A+BK)W\} & W & (A_\delta + BK) & \varsigma B - Z' \\ * & -\bar{Q}_x & 0 & 0 \\ * & * & -Q_\delta & -G_2' \\ * & * & * & -2\varsigma \end{bmatrix} < 0 \tag{7}$$

$$\begin{bmatrix} W & 0 & WK' - Z' & W \\ * & Q_\delta & K' - G_2' & 0 \\ * & * & u_0^2 & 0 \\ * & * & * & \bar{Q}_x \end{bmatrix} > 0 \tag{8}$$

then, $\mathcal{E}(W^{-1}) = \{x \in \mathbb{R}^{n+1};\ x'W^{-1}x \leq 1\}$ is a RAS for the system (3) under the sampling strategy given by Algorithm 1, with $Q_x = \bar{Q}_x^{-1}$.

Proof: Considering $\delta_p(t) \triangleq x_p(t_k) - x_p(t)$, $\delta_c(t) \triangleq x_c(t_k) - x_c(t)$ and $\mathrm{sat}(\nu) = \nu + \phi(\nu)$, note that between two trigger events, that is for $t \in (t_k, t_{k+1})$, we can re-write (3) as follows:

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t) \\ \dot{x}_c(t) = -C_p x_p(t) - C_p \delta_p(t) \\ u(t) = -k_p C_p[x_p(t) + \delta_p(t)] + k_i[x_c(t) + \delta_c(t)] \\ \qquad\qquad + \phi(\nu(t_k)) \\ y(t) = C_p x_p(t) \end{cases} \tag{9}$$

Using now A, B and K as defined in (6) and considering $x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix}$, $\delta \triangleq \begin{bmatrix} \delta_p \\ \delta_c \end{bmatrix}$ and $C \triangleq \begin{bmatrix} C_p & 0 \end{bmatrix}$, we can rewrite (9) as follows, where we dropped the time dependence to simplify the notation:

$$\begin{cases} \dot{x} = (A+BK)x + (A_\delta + BK)\delta + B\phi(K(x+\delta)) \\ y = Cx \end{cases} \tag{10}$$

Considering a quadratic Lyapunov function candidate $V(x(t)) = x'(t)Px(t)$, it follows that

$$\begin{aligned}\dot{V}(x) = x' \operatorname{He}\{P(A+BK)\}x \\ + 2x'P(A_\delta + BK)\delta + 2x'PB\phi\end{aligned} \tag{11}$$

Notice that the argument of ϕ, i. e. $\nu(t_k)$, can be written as

$$K(x+\delta) = \begin{bmatrix} K & K \end{bmatrix} \begin{bmatrix} x \\ \delta \end{bmatrix} \triangleq K_a x_a \tag{12}$$

where $K_a \triangleq \begin{bmatrix} K & K \end{bmatrix}$ and $x_a \triangleq \begin{bmatrix} x \\ \delta \end{bmatrix}$.

Hence, provided that x_a belongs to the region $\mathcal{S}_0 = \{x_a \in \mathbb{R}^{2(n+1)};\ \left|K_a x_a - G_a x_a\right| \leq u_0\}$, the following generalized sector condition ([16], [19]) is verified with respect to the deadzone nonlinearity ϕ:

$$\tau\phi'(K_a x_a)(\phi(K_a x_a) + G_a x_a) \leq 0 \tag{13}$$

with τ being a positive scalar and $G_a = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ a free matrix of appropriate dimensions.

Assuming $x_a \in \mathcal{S}_0$ and applying (13), we can write

$$\dot{V}(x) \leq \dot{V}(x) - 2\tau\phi'(K_a x_a)\big(\phi(K_a x_a) + G_a x_a\big) \tag{14}$$

Thus, from (11) and (12), (14) can be written as follows:

$$\dot{V}(x) \leq \begin{bmatrix} x' & \delta' & \phi' \end{bmatrix} M \begin{bmatrix} x' & \delta' & \phi' \end{bmatrix}' \tag{15}$$

with

$$M = \begin{bmatrix} \operatorname{He}\{P(A+BK)\} & P(A_\delta + BK) & PB - \tau G_1' \\ * & 0 & -\tau G_2' \\ * & * & -2\tau \end{bmatrix}$$

From Algorithm 1, it follows that $\delta' Q_\delta \delta - x' Q_x x \leq 0$. Then, if the following matrix inequality is verified:

$$\begin{bmatrix} \operatorname{He}\{P(A+BK)\} + Q_x & P(A_\delta + BK) & PB - \tau G_1' \\ * & -Q_\delta & -\tau G_2' \\ * & * & -2\tau \end{bmatrix} < 0 \tag{16}$$

from (15), we can conclude that $\dot{V}(x) < \delta' Q_\delta \delta - x' Q_x x \leq 0$ for $t \in (t_k, t_{k+1})$, provided $x_a \in \mathcal{S}_0$. Now pre- and post-multiplying (16) by $\operatorname{diag}(W, I, \varsigma)$ with $W = P^{-1}$, $\varsigma = \tau^{-1}$, considering the variable changes $Z = G_1 W$ and then applying Schur's complement, we retrieve matrix inequality (7), with $\bar{Q}_x = Q_x^{-1}$. Thus, the satisfaction of (7) guarantees $\dot{V}(x) < 0$ for $t \in (t_k, t_{k+1})$, provided $x_a \in \mathcal{S}_0$.

At the instants $t = t_k$, $\delta(t)$ is zero and the system reduces to the continuous-time (4). In this case, it can be verified that (16) implies (5) and it follows that $\dot{V}(x(t_k)) < 0$, provided $x_a(t_k) \in \mathcal{S}_0$.

Now we show that (8) guarantees that $x_a \in \mathcal{S}_0, \forall t \geq 0$, provided $x(0) \in \mathcal{E}(P)$. Pre- and post-multiplying (8) by $\operatorname{diag}(P, I, 1, I)$, applying Schur's complement twice and recalling that $Z = G_1 W$, the following relation is verified

$$\begin{bmatrix} P - Q_x & 0 \\ 0 & Q_\delta \end{bmatrix} - [K - G_1\ K - G_2]' \frac{1}{u_0^2} [K - G_1\ K - G_2] > 0 \tag{17}$$

Pre- and post-multiplying (17) by $\begin{bmatrix} x' & \delta' \end{bmatrix}$ and $\begin{bmatrix} x \\ \delta \end{bmatrix}$ respectively, (17) implies that

$$\begin{aligned} & x'Px + \delta' Q_\delta \delta - x' Q_x x \\ & - x_a'[K - G_1\ K - G_2]' \frac{1}{u_0^2} [K - G_1\ K - G_2] x_a > 0 \end{aligned} \tag{18}$$

Hence, if $t \in (t_k, t_{k+1})$, since $\delta' Q_\delta \delta - x' Q_x x \leq 0$, we conclude that $x_a(t) \in S_0$ if $x(t) \in \mathcal{E}(P)$. On the other hand, at $t = t_k$, we have $\delta(t) = 0$ and it also follows that $x_a(t_k) \in \mathcal{S}_0$ if $x(t_k) \in \mathcal{E}(P)$. Thus, we can conclude that if $x(0) \in \mathcal{E}(P)$, (7) along with (8) effectively ensures that $x_a(t) \in \mathcal{S}_0$ and $\dot{V}(x) < 0,\ \forall t \geq 0$, which ensures that $\mathcal{E}(P)$ is an invariant and contractive set with respect to the system (3), being included in its region of attraction. ■

Theorem 1 ensures the regional asymptotic stability of the closed-loop system. Actually, it can be applied to both stable or unstable plants. Nevertheless, if A_p is Hurwitz, global stability can be achievable [10]. A sufficient condition for the global stability of the closed-loop system under the event-trigger strategy can therefore be stated as a corollary to Theorem 1 as follows.

Corollary 1: If there exist a scalar $\varsigma > 0$, matrices $W = W' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$, $\bar{Q}_x = \bar{Q}_x' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$ and $Q_\delta = Q_\delta' > 0 \in \mathbb{R}^{(n+1)\times(n+1)}$ such that the following LMI is satisfied:

$$\begin{bmatrix} \operatorname{He}\{(A+BK)W\} & W & (A_\delta + BK) & \varsigma B - WK' \\ * & -\bar{Q}_x & 0 & 0 \\ * & * & -Q_\delta & -K' \\ * & * & * & -2\varsigma \end{bmatrix} < 0 \tag{19}$$

then, the origin of system (3) is globally asymptotically stable under the sampling strategy given by Algorithm 1, with $Q_x = \bar{Q}_x^{-1}$.

Proof: It follows the same lines as the proof of Theorem 1 but applying a global generalized sector condition, i. e., using $G_a = K_a$, which implies $G_1 = G_2 = K$. ■

Remark 1: It is worth noticing that the possibility of Zeno solutions is eliminated taking into account that the event-trigger function satisfies:

$$||\delta(t)|| \leq \frac{\lambda_{max}(Q_x)}{\lambda_{min}(Q_\delta)} ||x(t)|| \tag{20}$$

Hence, the same arguments used in [2] can be applied here to show that the inter-sampling times are lower bounded.

Remark 2: Notice that the plant and controller states need to be available to implement the event trigger criterion defined by Algorithm 1. Considering that the controller and the event generator run in different nodes, the controller state can be transmitted to the event generator only at the instants t_k. Then, assuming the controller sends its state at instants t_k, $x_c(t)$ can be recovered, as in [7], using:

$$x_c(t) = x_c(t_k) - \int_{t_k}^{t} y(t_k)dt = x_c(t_k) - y(t_k)(t - t_k) \tag{21}$$

In numerous applications only part of the plant state is measured. In this case, since $y(t)$ is assumed to be continuously

available to the event generator, $x_p(t)$ can be recovered through a Luenberger observer:

$$\hat{x}_p(t) = A_p\hat{x}_p + B_p u(t) + L(C_p\hat{x}_p - y(t)) \qquad (22)$$

where $\hat{x}_p$ is the estimated state.

Remark 3: In [14] and [15] it is assumed that the controller output is continuously applied to the plant. In this scenario, the closed-loop system is given by:

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t) \\ y(t) = C_p x_p(t) \\ \dot{x}_c(t) = -y(t_k) \\ u(t) = \mathrm{sat}(\nu(t)) \\ \nu(t) = k_i x_c(t) - k_p y(t_k) \quad \forall t \in [t_k, t_{k+1}) \end{cases} \qquad (23)$$

Our approach can be easily adapted to cope with this case, i.e. similar conditions to (7) and (8) can be obtained from the same steps used in the proof of Theorem 1. Particularly in this case, the trigger function can be defined only in terms of the plant and controller outputs. However, the possibility of Zeno solutions occurrence has to be carefully studied.

V. Optimization Problems

Consider a given region of admissible initial states $\mathcal{X}_0$. If $\mathcal{X}_0 \subset \mathcal{E}(P)$ and conditions of Theorem 1 are satisfied, we conclude that $\mathcal{X}_0$ is also included in the region of attraction of the closed-loop system. To solve Problem 1, the idea is also to reduce the number of transmission events, i. e., reduce the number of sampling instants.

As observed in Section IV, in order to reduce the sampling activity, one should aim at finding Q_x as "large" as possible and Q_δ as "small" as possible, while ensuring that the closed-loop system under the event-trigger strategy given in Algorithm 1 is stable. Considering that $\mathcal{X}_0$ is specified as an ellipsoid $\mathcal{E}(P_0) = \{x \in \mathbb{R}^{(n+1)};\quad x'P_0x \leq 1\}$, this goal can be achieved from the following convex optimization problem:

$$\begin{aligned} &\min(\mathrm{tr}(Q_\delta + \bar{Q}_x)) \\ &\text{subject to: (7), (8)}, \begin{bmatrix} W & I \\ I & P_0 \end{bmatrix} > 0 \end{aligned} \qquad (24)$$

Note that the last constraint in (24) ensures that $\mathcal{X}_0 \subset \mathcal{E}(P)$. Also note that this optimization problem, although leading to a reduction in the number of events, does not guarantee that the minimum possible is achieved.

VI. Numerical Example

Consider the following unstable plant:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{sat}(u(t)) \\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{cases} \qquad (25)$$

We choose controller gains $k_p = 18$ and $k_i = 19$ so that the system without saturation and with a continuous-time controller is stabilized with closed-loop poles in -10.2, -6.68 and -1.11.

We consider $\mathcal{X}_0 = \mathcal{E}(P_0)$, with $P_0 = \mathrm{diag}(4, 4, 10^6)$. Then we apply the optimization problem (24) proposed in the previous section, with an additional constraint $Q_\delta \leq 0.01I$ in order to prevent Q_δ from being ill-conditioned, obtaining:

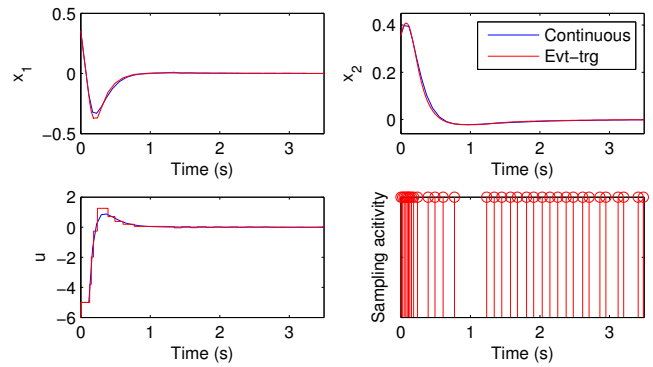

Fig. 2. Simulation results – Comparison between continuous and event-triggered systems

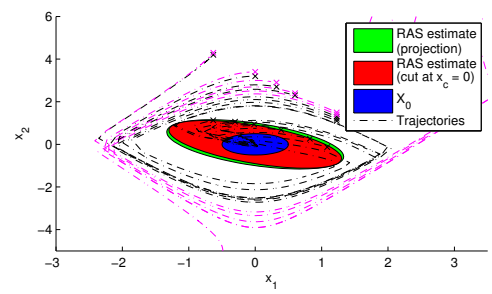

Fig. 3. RAS with event-triggered control

$$Q_x = \begin{bmatrix} 0.2360 & -0.1015 & -0.05127 \\ * & 0.4291 & 0.01756 \\ * & * & 1.643 \end{bmatrix}$$

$$Q_\delta = \begin{bmatrix} 11.33 & 11.32 & -10.72 \\ * & 11.33 & -10.72 \\ * & * & 12.49 \end{bmatrix} \quad P = \begin{bmatrix} 0.9049 & 0.6176 & -0.8064 \\ * & 1.219 & -0.9665 \\ * & * & 4.240 \end{bmatrix}$$

Figure 2 shows results of a simulation with $x(0) = [\sqrt{2}/4 \;\; \sqrt{2}/4 \;\; 0]'$. Only 40 trigger events are needed in the time interval $[0, 5]$ and the minimum inter-sampling time is 19 ms. For comparison purposes, the response considering a continuous-time implementation is also depicted in this figure. We can observe that there is no significant degradation on the system performance when compared to the continuous case. As a matter of comparison, if one uses a periodic sampled control law with a sampling period that achieves the same number of samples as the event-triggered, i. e. a sampling period around 100 ms, the trajectory for the same initial condition diverges. Figure 3 shows the projection of the set $\mathcal{E}(P)$ obtained for the event-triggered controller along with the projections of some convergent and some divergent trajectories, in black and in magenta, respectively. Note that $\mathcal{E}(P)$ contains $\mathcal{X}_0$, as required.

We consider now a more stringent specification for $\mathcal{X}_0$, by choosing $P_0 = \mathrm{diag}(1, 1, 10^6)$. Figure 4 shows the projection of the set $\mathcal{E}(P)$ obtained. One can see that $\mathcal{E}(P)$ now is closer to $\mathcal{X}_0$. Figure 5 shows results of a simulation with $x_0 = [\sqrt{2}/2 \;\; \sqrt{2}/2 \;\; 0]'$. Now 48 trigger events are needed in

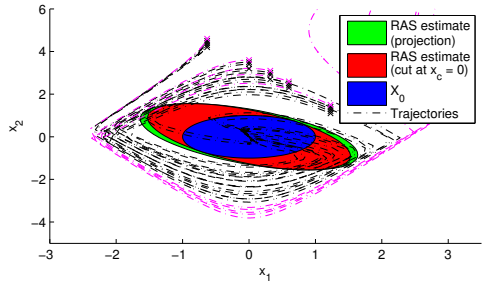

Fig. 4. RAS with event-triggered control – Large $\mathcal{X}_0$

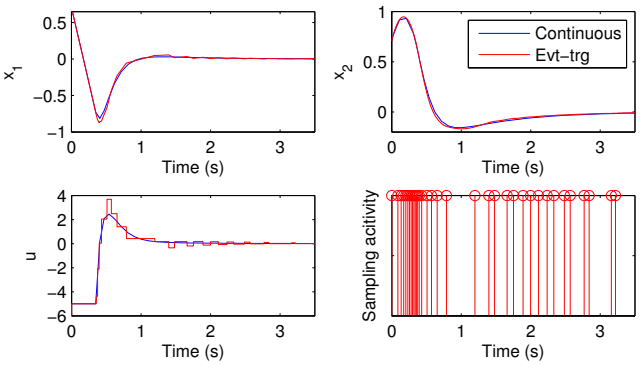

Fig. 5. Simulation results – Comparison between continuous and event-triggered systems – Large $\mathcal{X}_0$

the time interval $[0, 5]$ and the minimum inter-sampling time is 17 ms; a slight worse performance than the less restrictive $\mathcal{X}_0$ specification. One can also observe a slight degradation on the system performance when compared to the continuous case.

VII. Concluding Remarks

In this paper we proposed a systematic methodology to design event-triggered strategies for PI controlled plants subject to input saturation. Differently from previous work, we suppose that plant and controller are in different nodes in a network and that the value of the plant input is kept constant between two trigger instants, i.e. when the trigger condition is verified the plant output is transmitted to the controller and an updated control signal is transmitted to the plant. The method guarantees local asymptotic stability of the origin for a given set of initial conditions and can be easily extended to guarantee global stability for stable plants. Since the derived conditions are in an LMI form, a convex optimization problem is proposed to compute the trigger function parameters aiming at reducing the sampling activity while guaranteeing the stability of the closed-loop system.

The extension of the proposed approach to include anti-windup compensation, the co-design of the trigger function and controller parameters and incorporating the ideas from [20] to handle periodic event-triggered control are subjects of ongoing work.

References

[1] W. Heemels, K. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *51st IEEE Conference on Decision and Control (CDC)*, Dec 2012, pp. 3270–3285.

[2] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, Sept 2007.

[3] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nesic, "Co-design of output feedback laws and event-triggering conditions for linear systems," in *53rd IEEE Conference on Decision and Control (CDC)*, Dec 2014, pp. 3560–3565.

[4] P. Tallapragada and N. Chopra, "Decentralized event-triggering for control of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 12, pp. 3312–3324, 2014.

[5] S. Durand and N. Marchand, "Further results on event-based PID controller," in *2009 European Control Conference (ECC)*, Aug 2009, pp. 1979–1984.

[6] ——, "An Event-Based PID Controller With Low Computational Cost," in *8th International Conference on Sampling Theory and Applications (SampTA'09)*, L. Fesquet and B. Torrésani, Eds., Marseille, France, May 2009, p. Special session on Sampling and Industrial Applications.

[7] J. M. Gomes Da Silva Jr., W. F. Lages, and D. G. Sbarbaro, "Event-triggered PI control design," in *19th IFAC World Congress (IFAC WC 2014)*, Cape Town, South Africa, 2014, pp. 6947–6952.

[8] U. Tiberi, J. Araujo, and K. H. Johansson, "On event-based PI control of first-order processes," in *2nd IFAC Conference on Advances in PID Control, PID'12*, Brescia, Italy, Mar 2012, pp. 448–453.

[9] M. Beschi, S. Dormido, J. Sánchez, and A. Visioli, "Event-based PI controller with exponential thresholds," in *19th IFAC World Congress (IFAC WC 2014)*, Cape Town, South Africa, Aug. 2014, pp. 5766–5771.

[10] H. Sussmann, E. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2411–2425, Dec 1994.

[11] A. Seuret, C. Prieur, S. Tarbouriech, and L. Zaccarian, "Event-triggered control with LQ optimality guarantees for saturated linear systems," in *9th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2013)*, Toulouse, France, Sep. 2013, p. 6p.

[12] W. Wu, S. Reimann, and S. Liu, "Event-triggered control for linear systems subject to actuator saturation," in *19th IFAC World Congress (IFAC WC 2014)*, Cape Town, South Africa, Aug. 2014.

[13] D. Lehmann and K. H. Johansson, "Event-triggered PI control subject to actuator saturation," in *2nd IFAC Conference on Advances in PID Control, PID'12*, Brescia, Italy, pp. 430–435.

[14] D. Lehmann, G. Kiener, and K. Johansson, "Event-triggered PI control: Saturating actuators and anti-windup compensation," in *51st IEEE Conference on Decision and Control (CDC)*, Dec 2012, pp. 6566–6571.

[15] G. Kiener, D. Lehmann, and K. Johansson, "Actuator saturation and anti-windup compensation in event-triggered control," *Discrete Event Dynamic Systems*, vol. 24, no. 2, pp. 173–197, 2014.

[16] S. Tarbouriech, G. Garcia, J. M. Gomes Da Silva Jr., and I. Queinnec, *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, 2011.

[17] H. K. Khalil, *Nonlinear systems*. Upper Saddle River, (N.J.): Prentice Hall, 1996.

[18] R. Genesio, M. Tartaglia, and A. Vicino, "On the estimation of asymptotic stability regions: State of the art and new proposals," *IEEE Transactions on Automatic Control*, vol. 30, no. 8, pp. 747–755, 1985.

[19] J. M. Gomes Da Silva Jr. and S. Tarbouriech, "Antiwindup design with guaranteed regions of stability: an LMI-based approach," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 106–111, Jan 2005.

[20] E. Aranda-Escolastico, M. Guinaldo, and S. Dormido, "A novel approach for periodic event-triggering based on general quadratic functions," in *2015 International Conference on Event-based Control, Communication, and Signal Processing (EBCCSP)*, Krakow, Poland, June 2015, pp. 1–6.