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The Vehicle Routing Problem with Transhipment Facilities

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Abstract

This paper proposes an exact method for solving an optimization problem arising in several distribution networks, where customers can be served either directly, using vehicle routes from a central depot, or through transhipment facilities. The problem consists of optimizing the following interdependent decisions: selecting transhipment facilities, allocating customers to these facilities and designing vehicle routes emanating from a central depot to minimize the total distribution cost. This problem is called the Vehicle Routing Problem with Transhipment Facilities (VRPTF). The paper describes two integer programming formulations for the VRPTF, an edge-flow based formulation and a Set Partitioning (SP) based formulation. The LP-relaxation of the two formulations are further strengthened with the addition of different valid inequalities. Moreover, two new route relaxations that are used by dual ascent heuristics to find near-optimal dual solutions of the LP-relaxation of the SP model are described. The valid inequalities and the route relaxations are used in a branch-and-cut-and-price approach to solve the problem to optimality. The proposed method is tested on a large family of instances, including real-world instances, and the computational results obtained indicate the effectiveness of the proposed method.

Keywords: *transhipment facilities, dual ascent heuristic, column-and-cut generation*

1 Introduction

In several distribution networks the shipment to a customer is performed either directly, using vehicle routes emanating from a central depot, or through intermediate depots or *transhipment facilities*. In the latter case, the shipment is first delivered to a transhipment facility by a vehicle route, and then it is successively delivered to the final customer. Transhipment facilities provide a way to consolidate shipments into large vehicle loads, thereby allowing for a reduction of total distribution cost, and provide the capability to transfer shipments between different vehicles or modes of transportation (e.g., railroads, aircraft). In some cases, the transhipment facilities can be part of the same company which owns the central depot, and which makes the final delivery to the customers with its fleet of vehicles. In other cases, transhipment facilities are owned by a third-party subcontractor, who is also in charge of performing the final shipment to the customers.

The problem addressed in this paper is motivated by a real application of interest to an Italian company operating in the production and distribution of non-perishable products. More specifically, the problem consists of selecting transhipment facilities, allocating customers to these facilities and designing vehicle routes to minimize the total distribution cost. We call this problem the Vehicle Routing Problem with Transhipment Facilities (VRPTF). In the VRPTF, each customer can be served either directly by a vehicle route or through a facility selected from a set of potential facilities to which the customer can be assigned. The total load of a vehicle route, computed as the sum of the customer demands and of the quantities delivered to the facilities, must be less than or equal to the vehicle capacity. The problem objective is to minimize the total sum of routing and assignment costs.

1.1 Literature review

The VRPTF generalizes the well-known Capacitated Vehicle Routing Problem (CVRP). In the CVRP, a fleet of identical vehicles located at a central depot has to be optimally routed to supply a set of customers with known demands. Each vehicle performs at most one route, each customer must be visited exactly once, and the total demand of the customers visited by a route cannot exceed the vehicle capacity. The book edited by Toth and Vigo (2014) provides a comprehensive overview of exact methods for the CVRP and other variants.

As far as the authors know, the VRPTF has never been addressed in the literature. Closely related problems to the VRPTF are the Capacitated m-Ring-Star Problem (CmRSP), the Multiple Vehicle Traveling Purchaser Problem (MVTPP), the Two-Echelon Capacitated Vehicle Routing Problem (2E-CVRP), and the Location Routing Problem (LRP). The CmRSP, introduced in Baldacci et al. (2007), arises in the design of urban optical telecommunication networks and it consists of designing a set of rings that pass through a telephone exchange and through some transition points (also called *steiner nodes*) and/or users. Each nonvisited user must be assigned to a visited point or to a user. The number of users visited and assigned to a ring is bounded by the capacity of the ring. The objective is to minimize the total routing cost plus the assignment costs. The special case of the CmRSP arising when the users can be assigned only to steiner nodes, can be solved as a VRPTF with unit demands. The MVTPP described by Riera-Ledesma and Salazar-Gonzáï£ · lez (2012) models a family of routing problems combining stop selection and bus route generation. The problem consists of choosing a set of bus stops to which users are assigned, and simultaneously designing bus routes visiting such stops. The total number of users assigned to the stops of a route cannot exceed the seat capacity of a bus. The objective is to minimize the total length of all routes plus the total assignment cost. The undirected version of the MVTPP is equivalent to the VRPTF with the additional constraint imposing that the customers can only be assigned to facilities (or bus stops) and cannot be visited by a route. Both Baldacci et al. (2007) and Riera-Ledesma and Salazar-Gonzáï£ · lez (2012) proposed branch-and-cut approaches for the solution of the CmRSP and MVTPP, respectively. Recently, Riera-Ledesma and González (2013) also proposed a branch-and-cut-and-price algorithm for the MVTPP. The 2E-CVRP is a two-level distribution system where the deliveries to customers from a depot are managed through intermediate capacitated depots, called satellites. The first level consists of vehicle routes visiting satellites only whereas the second level routes supply all customers. The main difference between the VRPTF and the 2E-CVRP is that in the VRPTF a customer can be either visited on a route or assigned to a facility, whereas in the 2E-CVRP each customer is visited once by exactly a second level route. The 2E-CVRP model is particularly useful when the facilities are part of the same company owing the main depot whereas in the VRPTF model the facilities are generally owned by third-party contractors, which are in charge of delivering to the final customers the quantity consolidated at the facilities. Exact methods for the 2E-CVRP have been proposed by Jepsen et al. (2013) and Baldacci et al. (2013). The LRP is a special case of the 2E-CVRP and consists of opening a set of depots and designing a set of routes for each opened depot so that the total load of the routes operated from a depot does not exceed its capacity and each customer is visited by exactly one route. The objective is to minimize the sum of the fixed costs of the opened depots and the costs of the routes operated from the depots. A recent review of location routing problem variants and heuristic and exact algorithms can be found in Prodhon and Prins (2014).

Another related problem to the VRPTF is the Multi-Vehicle Covering Tour Problem (m-CTP) introduced by Hachicha et al. (2000). In the m-CTP two sets of locations are given. The first set, consists of potential locations at which some vehicles may stop, and the second set are locations not actually on vehicle routes, but within an acceptable distance from a vehicle route. The m-CTP consists of determining a set of total minimum length vehicle routes on a subset of the first set of locations, subject to side constraints, such that every location of the second set is within a prespecified distance from a route. Há et al. (2013) proposed a branch-and-cut for the variant named the m-CTP-p where an upper bound on the number of vertices per route is given with a parameter p and the m number of vehicles used is a decision variable. The m-CTP-p is equivalent to the VRPTF with the additional constraint imposing that the customers can only be assigned to facilities and cannot be visited by a route, and that all assignment costs are equal to zero.

The VRPTF does not require any specific synchronization of incoming and outgoing vehicles at the facilities. In some practical applications, a correct synchronization can be required and in this case the facilities are generally referred as cross-docking facilities. For an overview of the cross-docking concept and extensive review of the existing literature the reader is referred to Belle et al. (2012). In this context, a generic class of VRPs that has recently received attention in the literature is the class of VRPs with Multiple Synchronization Constraints (VRPMSs). VRPMSs exhibit synchronization requirements between the vehicles, concerning spatial, temporal, and load aspects. A review of VRPMS presenting a classification of different types of synchronizations and a discussion about heuristic and exact algorithms can be found in Drexl (2012).

1.2 Contributions of this paper

This paper addresses a new problem of practical relevance and proposes both heuristic and exact methods for its solution. More specifically, we introduce a *two-index* formulation (TI) and we describe different valid inequalities for it, both by adapting those already proposed for the CmRSP, and by introducing new ones specific for the VRPTF. We also describe lower bounds derived from a set-partitioning based formulation (SP) of the problem, and computed using two efficient dual ascent heuristics that use two new route relaxations, called q-$*$route and ng-$*$route, respectively. The proposed methods have been tested on a large family of instances, including both instances derived from the literature and real-world instances. The computational results show that real-world instances with up to 142 customers and 18 facilities were solved to optimality and that high quality solutions were computed for instances with up to 164 customers. In addition, tight lower bounds were computed, with average percentage deviations equal to 98.7% and 97.4% for real-word and literature-based instances, respectively.

This paper is organized as follows. The next section formally introduces the VRPTF and presents formulation TI for which different valid inequalities are described in Section 3. Section 4 presents formulation SP and lower bounds based on its LP-relaxation; some properties of the LP-relaxation of SP are also investigated in the section. A bounding method used to compute a lower bound on the VRPTF is described in Section 5. Section 6 describes the exact method used to solve the VRPTF to optimality together with two heuristic algorithms. Section 7 reports computational results, and concluding remarks are given in Section 8.

2 Problem description and Two-Index (TI) formulation

This section describes the VRPTF and presents a *edge-flow* based formulation to model it.

The VRPTF is defined on a mixed graph $G = (V, E \cup A)$, where $V = \{0\} \cup V'$ is the node set, $E = \{\{i, j\} : i, j \in V, i \neq j\}$ is the edge set, and A is the arc set. Node set V' is partitioned into two subsets: $V_C = \{1, \dots, n_C\}$ containing a node for each customer and $V_F = \{n_C + 1, \dots, n_C + n_F\}$ containing a node for each transhipment facility. Node 0 represents a central depot. Each customer $i \in V_C$ requires a supply of q_i units from the depot (we assume $q_i = 0$, $\forall i \in \{0\} \cup V_F$) that can be delivered either directly from a vehicle route emanating from the depot or through a facility selected from a set $F_i \subseteq V_F$ of facilities to which customer i can be assigned. Set A represents the possible assignments between customers and facilities, i.e., $A = \{(i, j) : i \in V_C, j \in F_i\}$. Set E is the set of possible route edges, each edge $e = \{i, j\} \in E$ is associated with a non-negative *routing* cost $r_e = r_{\{i,j\}}$, while each arc $(i, j) \in A$ is associated with a non-negative *assignment* cost d_{ij}. Henceforth, if e connects the two nodes i and j then $\{i, j\}$ and e will be used interchangeably to denote the same edge.

A *route* is defined by a pair (R, A') where $R = (0, i_1, \dots, i_r, 0)$, $r \geq 1$, is a simple cycle in G passing through the depot, visiting nodes $V(R) = \{i_1, \dots, i_r\} \subseteq V'$, and $A' \subseteq A$ are assignments between customers of $V_C \setminus V(R)$ and nodes of $V(R) \cap V_F$. Notice that if $r = 1$ then route R represents the single-node route $R = (0, i_1, 0)$. We say that a customer i is *assigned* to a route R if it is either visited by the simple cycle (i.e., $i \in V(R)$) or it is connected to a node of the route representing a facility (i.e., a node $j \in V(R) \cap V_F$ exists such that $(i, j) \in A'$). The total load of a route is computed as the sum of the demands of the customers assigned to the route. The route is *feasible* if its total load does not exceed the vehicle capacity Q. The *cost* of a route is equal to the sum of the routing costs of the edges forming the route plus the sum of the assignment costs of the arcs in A'.

The aim of VRPTF is to design a set of routes so that each customer is assigned to exactly one route, each intermediate facility is visited at most once and the sum of the route costs is minimized.

We will use the following notation throughout. For any $S \subseteq V'$, let $V_C(S) = S \cap V_C$ and $V_F(S) = S \cap V_F$ denote the set of customers and of facilities in S, respectively. Let $F_i(S) = V_F(S) \cap F_i$ denote the set of facilities in S associated with customer $i \in V_C$. Also for any set $S \subseteq V$, define $\delta(S) = \{\{i, j\} \in E : i \in S, j \notin S\}$ (if $S = \{i\}$, we simply write $\delta(i)$ instead of $\delta(\{i\})$).

Let x_e be an integer variable which takes value in $\{0, 1\}$, $\forall e \in E \setminus \{\{0, j\} : j \in V'\}$ and value in $\{0, 1, 2\}$, $\forall e \in \{\{0, j\} : j \in V'\}$. Notice that $x_{\{0,j\}} = 2$ when the single-node cycle $R = (0, j, 0)$ is selected in the solution. For each arc $(i, j) \in A$, let z_{ij} be a binary variable which is equal to 1 if and only if customer i is assigned to node j. Moreover, for each $i \in V'$, let y_i be a binary variable which is equal to 1 if and only if node i is on a route. Formulation TI is as follows:

$$(TI) \quad \min \sum_{e \in E} r_e x_e + \sum_{(i,j) \in A} d_{ij} z_{ij} \tag{1}$$

$$s.t. \sum_{e \in \delta(i)} x_e = 2y_i, \qquad \forall i \in V' \tag{2}$$

$$y_i + \sum_{j \in F_i} z_{ij} = 1, \qquad \forall i \in V_C \tag{3}$$

$$\sum_{e \in \delta(S)} x_e \geq \frac{2}{Q} \left(\sum_{i \in V_C(S)} q_i y_i + \sum_{(i,j) \in A: j \in V_F(S)} q_i z_{ij} \right), \qquad \forall S \subseteq V' : S \neq \emptyset \tag{4}$$

$$x_e \in \{0,1\}, \qquad \forall e \in E \setminus \{\{0,j\} : j \in V'\} \tag{5}$$

$$x_e \in \{0,1,2\}, \qquad \forall e \in \{\{0,j\} : j \in V'\} \tag{6}$$

$$z_{ij} \in \{0,1\}, \qquad \forall (i,j) \in A \tag{7}$$

$$y_i \in \{0,1\}, \qquad \forall i \in V'. \tag{8}$$

Constraints (2) impose that the degree of each node $i \in V'$ is 2 if the node is on a route. Constraints (3) state that a customer $i \in V_C$ is either on a route or is assigned to one of its facilities. Inequalities (4) are the *fractional route capacity inequalities* (FrCC). These constraints, within the integrality of x, z and w variables, impose that for a given subset S of nodes, at least $\left\lceil (\sum_{i \in S} q_i y_i + \sum_{(i,j) \in A: j \in S} q_i z_{ij})/Q \right\rceil$ routes are needed to visit the customers assigned to nodes in S.

3 Strengthening the LP-relaxation of formulation TI

A number of valid inequalities can be used to improve the quality of the lower bound obtained from the LP-relaxation of formulation TI. In this section, we first derive valid inequalities by extending the results proposed for the CmRSP by Baldacci et al. (2007) to the VRPTF. Then, a new class of valid inequalities specifically devised for the VRPTF is introduced. The separation procedures for different valid inequalities are then described in Section 5.2.

Simple valid inequalities are the following: (i) $x_{\{i,j\}} \leq y_j, i \in V_C, j \in V_C, i \neq j$; (ii) $x_{\{i,j\}} \leq y_j, i \in V_F, j \in V', i \neq j$; (iii) $x_{\{i,j\}} + z_{ij} \leq y_j, i \in V_C, j \in F_i$, (iv) $y_j \leq \sum_{i \in V_C : j \in F_i} z_{ij}, \forall j \in V_F$. Further, the following inequalities are also valid.

a) *Connectivity inequalities* (CI):

$$\sum_{e \in \delta(S)} x_e \geq 2 \left(y_i + \sum_{j \in V_F(S) \cap F_i} z_{ij} \right), \quad \forall S \subseteq V', \forall i \in V_C(S), S \neq \emptyset. \tag{9}$$

b) *Multistar inequalities* (MI):

$$\sum_{e \in \delta(S)} x_e \geq \frac{2}{Q} \left(\sum_{i \in V_C(S)} q_i y_i + \sum_{(i,j) \in A: j \in V_F(S)} q_i z_{ij} + \sum_{i \in V_C(\overline{S})} \sum_{j \in S} q_i x_{\{i,j\}} \right), \forall S \subseteq V', S \neq \emptyset. \tag{10}$$

where $\overline{S} = V' \setminus S$.

c) *Rounded capacity constraints I* (RCI):

$$\sum_{e\in\delta(S)} x_e \geq 2\left\lceil \sum_{i\in S:F_i\subseteq S} q_i/Q \right\rceil, \quad \forall S\subseteq V', V_C(S)\neq\emptyset. \tag{11}$$

d) *Rounded capacity constraints II* (RCII):

$$\sum_{e\in\delta(S)} x_e \geq 2\left\lceil \left(\sum_{i\in V_C(S)} q_i y_i + \sum_{\substack{(i,j)\in A:\\ j\in V_F(S)}} q_i z_{ij} \right)/Q \right\rceil, \quad \forall S\subseteq V', S\neq\emptyset. \tag{12}$$

Notice that CI inequalities are not dominated by MI inequalities whereas MI inequalities dominate FrCC inequalities. RCII inequalities (12) are clearly nonlinear. In the next section, we describe two ways of linearizing inequalities (12). The first linearization extends to the VRPTF a similar linearization proposed for the C*m*RSP, whereas the second one is new and it is based on mixed integer optimization.

3.1 Linearized versions of inequalities RCII

A first family of valid inequalities can be obtained using the following lemma, proposed by Baldacci et al. (2007).

Lemma 1 *Let m, n and o be three non-negative integer values with $m > o$ and $mod(m, o) \neq 0$:*

$$\left\lceil \frac{m-n}{o} \right\rceil \geq \left\lceil \frac{m}{o} \right\rceil - \frac{n}{mod(m,o)}. \tag{13}$$

□

The term $\sum_{i\in V_C(S)} q_i y_i + \sum_{(i,j)\in A: j\in V_F(S)} q_i z_{ij}$ of RCII inequalities (12) can be rewritten as:

$$q(V_C) - \left(\sum_{i\in V_C(\overline{S})} q_i y_i + \sum_{(i,j)\in A: j\in V_F(\overline{S})} q_i z_{ij} \right) \tag{14}$$

and by using Lemma 1, from expression (14) we obtain the following inequality valid for any $S \subseteq V'$, $S \neq \emptyset$:

$$\sum_{e\in\delta(S)} \frac{1}{2} x_e \geq \left\lceil \frac{q(V_C)}{Q} \right\rceil - \frac{1}{mod(q(V_C),Q)} \left(\sum_{i\in V_C(\overline{S})} q_i y_i + \sum_{(i,j)\in A: j\in V_F(\overline{S})} q_i z_{ij} \right), \tag{15}$$

hereafter called RCII-a inequalities.

The term $\sum_{i\in V_C(S)} q_i y_i + \sum_{(i,j)\in A: j\in V_F(S)} q_i z_{ij}$ of RCII inequalities (12) can also be rewritten as:

$$q(V_C(S)) - \left(\sum_{\substack{(i,j)\in A:\\ i\in V_C(S), j\in V_F(\overline{S})}} q_i z_{ij} - \sum_{\substack{(i,j)\in A:\\ i\in V_C(\overline{S}), j\in V_F(S)}} q_i z_{ij} \right), \tag{16}$$

and by using Lemma 1 and by disregarding the term $\sum_{\substack{(i,j)\in A:\\ i\in V_C(\overline{S}),j\in V_F(S)}} q_i z_{ij}$ from (16) we get:

$$\sum_{e\in\delta(S)} \frac{1}{2}x_e \geq \left\lceil \frac{q(V_C(S))}{Q} \right\rceil - \frac{1}{mod(q(V_C(S)),Q)} \sum_{\substack{(i,j)\in A:\\ i\in V_C(S),j\in V_F(\overline{S})}} q_i z_{ij}, \tag{17}$$

hereafter called RCII-b inequalities.

Proposition 1 of the e-companion to this paper shows that there is no dominance between inequalities RCII-a and RCII-b.

The following lemma is based on mixed integer optimization. For a number $m \in \mathbb{R}$, define $\hat{m} = m - \lfloor m \rfloor$ to be its fractional part.

Lemma 2 *Let $o \in \mathbb{R}$ with $\hat{o} > 0$ and $T = \{m \in \mathbb{R}, n \in \mathbb{Z} : m + n \geq o, m \geq 0\}$. The following inequality is valid for T:*

$$m + \hat{o}n \geq \hat{o}\lceil o \rceil. \tag{18}$$

Proof. The proof is provided in the e-companion to this paper. □

Based on the above lemma, a second family of valid inequalities for the VRPTF can be obtained using the following theorem.

Theorem 1 *Let $\alpha_e \geq 0$, $\forall e \in E$, $\beta_i \geq 0$, $\forall i \in V'$ and $\gamma_{ij} \geq 0$, $\forall (i,j) \in A$ and consider the following inequality valid for formulation TI:*

$$\sum_{e\in E} \alpha_e x_e + \sum_{i\in V'} \beta_i y_i + \sum_{(i,j)\in A} \gamma_{ij} z_{ij} \geq o \tag{19}$$

where $o \in \mathbb{R}$ and $\hat{o} > 0$. Then the following inequality:

$$\sum_{e\in E} \varphi^o(\alpha_e)x_e + \sum_{i\in V'} \varphi^o(\beta_i)y_i + \sum_{(i,j)\in A} \varphi^o(\gamma_{ij})z_{ij} \geq \lceil o \rceil \tag{20}$$

where $\varphi^o(m) = \lfloor m \rfloor + \min\left\{\frac{\hat{m}}{\hat{o}}, 1\right\}$, $m \in \mathbb{R}$, $n \in \mathbb{R}$, $\hat{o} > 0$, is also a valid inequality for formulation TI.

Proof. The proof is provided in the e-companion to this paper. □

Notice that, inequality (19) can be scaled by a rational number t thus obtaining the following valid inequality for formulation TI:

$$\sum_{e\in E} \varphi^{to}(t\alpha_e)x_e + \sum_{i\in V'} \varphi^{to}(t\beta_i)y_i + \sum_{(i,j)\in A} \varphi^{to}(t\gamma_{ij})z_{ij} \geq \lceil to \rceil. \tag{21}$$

Starting from inequalities (4) and substituting the right-and side according to expressions (14) and (16) we get:

$$\sum_{e\in\delta(S)} \frac{1}{2}x_e + \sum_{i\in V_C(\overline{S})} \frac{q_i}{Q} y_i + \sum_{(i,j)\in A: j\in V_F(\overline{S})} \frac{q_i}{Q} z_{ij} \geq \frac{q(V_C)}{Q}, \tag{22}$$

and

$$\sum_{e\in\delta(S)} \frac{1}{2}x_e + \sum_{\substack{(i,j)\in A:\\ i\in V_C(S), j\in V_F(\overline{S})}} \frac{q_i}{Q} z_{ij} - \sum_{\substack{(i,j)\in A:\\ i\in V_C(\overline{S}), j\in V_F(S)}} \frac{q_i}{Q} z_{ij} \geq \frac{q(V_C(S))}{Q}. \tag{23}$$

First of all, notice that for $m, n \in \mathbb{R}$, $mod(m,n) = n((m/n) - \lfloor m/n \rfloor)$. Then, by setting $o = \frac{q(V_C)}{Q}$ and as $\varphi^o(\frac{1}{2}) = \min\left\{\frac{Q}{2mod(q(V_C),Q)}, 1\right\}$ and $\varphi^o(\frac{q_i}{Q}) = \min\left\{\frac{q_i}{mod(q(V_C),Q)}, 1\right\}$, $\forall i \in V_C$, from Theorem 1 and inequality (22) we obtain the following valid inequality:

$$\sum_{e\in\delta(S)} \min\left\{\frac{Q}{2\ mod(q(V_C),Q)}, 1\right\} x_e \geq \left\lceil \frac{q(V_C)}{Q} \right\rceil - \sum_{i\in V_C(\overline{S})} \min\left\{\frac{q_i}{mod(q(V_C),Q)}, 1\right\} y_i - \sum_{(i,j)\in A: j\in V_F(\overline{S})} \min\left\{\frac{q_i}{mod(q(V_C),Q)}, 1\right\} z_{ij}. \tag{24}$$

Also from Theorem 1, by disregarding the negative term of inequality (23) we obtain:

$$\sum_{e\in\delta(S)} \min\left\{\frac{Q}{2\ mod(q(V_C(S)),Q)}, 1\right\} x_e \geq \left\lceil \frac{q(V_C(S))}{Q} \right\rceil - \sum_{\substack{(i,j)\in A:\\ i\in V_C(S), j\in V_F(\overline{S})}} \min\left\{\frac{q_i}{mod(q(V_C(S)),Q)}, 1\right\} z_{ij}. \tag{25}$$

We call inequalities (24) and (25) RCII-c and RCII-d inequalities, respectively. Inequalities RCII-c and RCII-d are stronger than the pure integer rounding inequalities obtained from inequalities (22) and (23). In addition, notice that the coefficients of variables $\{x_e\}$ in both inequalities (24) and (25) are greater than 0.5 and less than or equal to 1. If $q_i = 1$, $\forall i \in V_C$, inequalities RCII-a and RCII-b dominate inequalities RCII-c and RCII-d. In general, no dominance relations exist among the four types of inequalities RCII-a, RCII-b, RCII-c and RCII-d.

4 Lower bounds based on a Set-Partitioning (SP) formulation

In this section, we first describe a Set-Partitioning (SP) based formulation for the VRPTF. Then, we investigate lower bounds based on the LP-relaxation of formulation SP. We introduce a theorem that is used to derive two dual ascent heuristics to find near-optimal dual solutions of the LP-relaxation of the SP model. Then, we describe how the valid inequalities described for the TI formulation in the previous sections can be used for strengthening the value of the LP-relaxation of formulation SP. Finally, we derive some properties of the LP-relaxation of formulation SP.

Let $\mathscr{R}$ be the index set of all feasible routes. Given a route $\ell \in \mathscr{R}$, we denote with R_ℓ the sequence $(i_1 = 0, i_2, \ldots, i_r = 0)$ of the nodes visited by the route and with $V_C(R_\ell)$ and $V_F(R_\ell)$ the sets $V_C \cap V(R_\ell)$ and $V_F \cap V(R_\ell)$, respectively. In addition, $V_A(R_\ell)$ denotes the customers of the route assigned to facilities in $V_F(R_\ell)$. Let $a_{i\ell}$ be a (0-1) binary coefficient equal to 1 if node $i \in V(R_\ell)$, 0 otherwise. In addition, let $b^j_{i\ell}$ be a (0-1) binary coefficient equal to 1 if customer $i \in V_A(R_\ell)$ is assigned to node $j \in V_F(R_\ell)$, 0 otherwise. Given a route ℓ, we denote with c_ℓ its routing cost computed as $\sum_{h=2}^{|R_\ell|} r_{\{i_{h-1},i_h\}}$, and with p_ℓ its assignment cost computed as $\sum_{j\in V_F(R_\ell)} \sum_{i\in V_A(R_\ell)} b^j_{i\ell} d_{ij}$. Let ξ_ℓ, $\ell \in \mathscr{R}$, be a (0-1) binary variable equal to 1 if and only if route ℓ is in the optimal solution.

Formulation SP is as follows:

$$(SP) \quad \min \sum_{\ell \in \mathscr{R}} (c_\ell + p_\ell)\xi_\ell \tag{26}$$

$$s.t. \sum_{\ell \in \mathscr{R}} \overline{a}_{i\ell}\xi_\ell = 1, \quad \forall i \in V_C \tag{27}$$

$$\sum_{\ell \in \mathscr{R}} a_{i\ell}\xi_\ell \leq 1, \quad \forall i \in V_F \tag{28}$$

$$\xi_\ell \in \{0,1\}, \qquad \forall \ell \in \mathscr{R}, \tag{29}$$

where $\overline{a}_{i\ell} = a_{i\ell} + \sum_{j \in V_F(R_\ell)} b^j_{i\ell}$, $i \in V_C$, $\ell \in \mathscr{R}$. In the formulation, constraints (27) and (28) impose that each customer is assigned exactly once and each facility is visited at most once, respectively.

We denote by LSP the LP-relaxation of formulation SP and by DSP the dual of LSP. The variables of DSP are given by the vector $\mathbf{u} = \{u_0, u_1, \ldots, u_{|V_C|}, u_{|V_C|+1}, \ldots, u_{|V'|}\}$, where $u_0 = 0$ for the depot, $u_1, \ldots, u_{|V_C|}$ are associated with constraints (27), and $u_{|V_C|+1}, \ldots, u_{|V'|}$, with constraints (28). In the following, we denote with $q^{min} = \min_{i \in V_C}\{q_i\}$. The following theorem holds.

Theorem 2 *Let us associate penalties $\lambda_i \in \mathbb{R}$, $\forall i \in V_C$, with constraints* (27)*, and $\lambda_i \leq 0$, $\forall i \in V_F$, with constraint* (28)*. Let $\mathscr{R}_i = \{\ell \in \mathscr{R} : \overline{a}_{i\ell} > 0\}$. For each $i \in V_C$ compute:*

$$\nu_i = q_i \min_{\ell \in \mathscr{R}_i} \left\{ \frac{(c_\ell + p_\ell) - \sum_{j \in V_C} \overline{a}_{j\ell}\lambda_j - \sum_{j \in V_F} a_{j\ell}\lambda_j}{\sum_{j \in V_C} \overline{a}_{j\ell} q_j} \right\}. \tag{30}$$

A feasible DSP solution $\mathbf{u}$ of cost $z(DSP(\boldsymbol{\lambda}))$ is given by the following expressions:

$$u_0 = 0 \quad \text{and } u_i = \nu_i + \lambda_i, \forall i \in V_C, \quad \text{and } u_i = \lambda_i, \forall i \in V_F. \tag{31}$$

Proof. The proof is provided in the e-companion to this paper. □

The pricing problem associated with formulation SP is a strongly $\mathcal{NP}$–hard problem, since it requires finding minimum cost elementary routes over a graph with both positive and negative edge and arc costs. In the special case where $V_F = \emptyset$, the pricing problem consists of finding capacitated elementary cycles, a strongly $\mathcal{NP}$-hard problem (see Poggi and Uchoa 2014).

Therefore, in practice we enlarge the set of routes $\mathscr{R}$ to contain also *nonnecessarily elementary* routes, i.e., coefficients $\overline{a}_{i\ell}$ are general nonnegative integer, thus a node can be visited in a route more than once and/or a customer can be assigned more than once to facilities of the routes. Although non-elementary routes are infeasible, this relaxation has the advantage that the pricing subproblem becomes solvable in pseudo-polynomial time (by dynamic programming). Moreover, Theorem 2 remains valid if the set of routes $\mathscr{R}$ is enlarged to contain also nonnecessarily elementary routes.

In Section 5, we introduce two route relaxations called q-∗route and ng-∗route, used by two dual ascent heuristics based on Theorem 2 to find near-optimal solutions of problem DSP. q-∗route and ng-∗route relaxations are based on route relaxations already proposed for the CVRP and on the observation that given a route $R_\ell = (i_1 = 0, i_2, \ldots, i_r = 0)$, a lower bound on its cost $c_\ell + p_\ell$ can be computed as $\sum_{h=2}^{|R_\ell|} r_{\{i_{h-1},i_h\}} + \sum_{j \in V_F(R_\ell)} lb_j$, where $lb_j \leq \sum_{i \in V_A(R_\ell)} b^j_{i\ell} d_{ij}$. Each value lb_j, $j \in V_F(R_\ell)$, can be computed as the minimum of the costs of all possible assignments of facility j involving customers in $\{i :\in V_C : j \in F_i\}$ with a total load $q = \sum_{i \in V_A(R_\ell)} b^j_{i\ell} q_i$.

Formulation LSP can be strengthened by adding valid inequalities derived for the TI formulation as follows. For each $\ell \in \mathscr{R}$, let coefficients η^ℓ_e be defined as follows: if ℓ is a route covering node h

only, then $\eta^\ell_{\{0,h\}} = 2$ and $\eta^\ell_{\{i,j\}} = 0$, $\forall \{i,j\} \in E \setminus \{0,h\}$; if ℓ is not a single-node route, then $\eta^\ell_{\{i,j\}} = 1$ for each edge $\{i,j\}$ traversed by route R_ℓ, and $\eta^\ell_{\{i,j\}} = 0$ otherwise.

Any feasible solution ξ of SP can be transformed into a feasible TI solution (x,z,w) by setting:

$$x_e = \sum_{\ell \in \mathscr{R}} \eta^\ell_e \xi_\ell, \quad \forall e \in E, \tag{32}$$

$$z_{ij} = \sum_{\ell \in \mathscr{R}} b^j_{i\ell} \xi_\ell, \quad \forall (i,j) \in A, \tag{33}$$

$$y_i = \sum_{\ell \in \mathscr{R}} a_{i\ell} \xi_\ell = 1 - \sum_{j \in F_i} \sum_{\ell \in \mathscr{R}} b^j_{i\ell} \xi_\ell, \quad \forall i \in V_C, \text{ and} \tag{34}$$

$$y_i = \sum_{\ell \in \mathscr{R}} a_{i\ell} \xi_\ell, \quad \forall i \in V_F. \tag{35}$$

The following theorem shows that any feasible solution of formulation LSP already satisfies some valid inequalities derived from formulation TI.

Theorem 3 *The LP-relaxation of the SP formulation satisfies both CI and FrCI inequalities, and a weak form of MI inequalities.*

Proof. The proof is provided in the e-companion to this paper. □

5 Bounding procedure

This section presents a method for computing a lower bound on the VRPTF which combines in sequence two dual ascent heuristics (see Section 5.1), and a column-and-cut generation method (see Section 5.2), all based on formulation LSP.

5.1 Dual ascent heuristics

The dual ascent heuristics are based on Theorem 2 where the set of routes $\mathscr{R}$ is enlarged with set $\mathscr{R}^>$ containing also nonnecessarily elementary routes (i.e., $\mathscr{R}^> \supseteq \mathscr{R}$). In particular, two different route relaxations are used, called q-∗route and ng-∗route, to compute lower bounds LB_1 and LB_2 on the VRPTF, respectively. The two dual ascent heuristics are based on a column generation-like method, called CG for solving the following problem:

$$LCG = \max_{\boldsymbol{\lambda}} \{z(DSP(\boldsymbol{\lambda}))\}. \tag{36}$$

CG executes a number of macro-iterations to compute a dual solution $\mathbf{u}$ of the master problem DSP, defined by the route subset $\overline{\mathscr{R}} \subseteq \mathscr{R}^>$, and then CG solves problem (36) with a predefined number $Maxit2$ of subgradient iterations to modify the penalty vector $\boldsymbol{\lambda}$.

5.1.1 Route relaxation q-$*$route

q-$*$routes are based on the q-path relaxation proposed by Christofides et al. (1981). We define a q-$*$path as a nonnecessarily elementary partial route in G from depot 0 to node $i \in V'$ with a load equal to q. In a q-$*$path a node $i \in V'$ can be visited more than once and a customer $i \in V_C$ can be assigned more than once. In the following, we describe a dynamic programming algorithm for computing q-$*$paths, with the restriction that a q-$*$path can not contain loops formed by three consecutive nodes. Let $f(q,i)$ be the cost of the least cost q-$*$path from node 0 to node i and let $\pi(q,i)$ be the node immediately before i in the least cost path of value $f(q,i)$. Let $g(q,i)$ be the cost of the least cost q-$*$path from node 0 to node i, such that $\gamma(q,i) \neq \pi(q,i)$, where $\gamma(q,i)$ is the node immediately before i in the least cost path corresponding to $g(q,i)$. For a given value of q, let $h(i,j)$ be the cost of the least cost q-$*$path from 0 to j, with $i \in V'$ just before j and without loops. In addition, for each facility $k \in V_F$, let $lb_k(q)$ be a lower bound on the assignment cost of any assignment of load q of customers to the facility k. $lb_k(q)$, for each $k \in V_F$ and $q^{min} \leq q \leq Q$, can be computed as the optimal solution cost of the following knapsack problem $KP(q,k)$:

$$(KP(q,k)) \quad lb_k(q) = \min \sum_{i \in V_C : k \in F_i} d_{ik}\chi_i \tag{37}$$

$$s.t. \sum_{i \in V_C : k \in F_i} q_i \chi_i = q \tag{38}$$

$$\chi_i \in \{0,1\}, \qquad \forall i \in V_C : k \in F_i. \tag{39}$$

We assume that $lb_k(q) = \infty$ if problem $KP(q,k)$ does not admit a feasible solution for the given pair q and k. For each $q = q^{min}, \ldots, Q$ and $i, j \in V'$, $i \neq j$, compute:

$$h(i,j) = \begin{cases} \begin{cases} f(q-q_j,i) + r_{\{i,j\}}, & \text{if } \pi(q-q_j,i) \neq j \\ g(q-q_j,i) + r_{\{i,j\}}, & \text{otherwise.} \end{cases} & , \ j \in V_C \\ \min_{q^{min} \leq w \leq Q} \begin{cases} f(q-w,i) + r_{\{i,j\}} + lb_j(w), & \text{if } \pi(q-w,i) \neq j \\ g(q-w,i) + r_{\{i,j\}} + lb_j(w), & \text{otherwise.} \end{cases} & , \ j \in V_F \end{cases} \tag{40}$$

Then, compute:

$$\begin{cases} f(q,j) = \min_{i \in V' \setminus \{j\}} \{h(i,j)\} \\ \pi(q,j) = i' \end{cases} \tag{41}$$

where i' is the node producing the above minimum,

$$\begin{cases} g(q,j) = \min_{i \in V' \setminus \{j,i'\}} \{h(i,j)\} \\ \gamma(q,j) = i'' \end{cases} \tag{42}$$

where i'' is the node producing the above minimum. The functions are initialized as follows:

- $f(q_j,j) = r_{0j}$, $\pi(q_j,j) = 0$, $j \in V_C$;
- $f(q,j) = \infty$, $\pi(q,j) = 0$, $q = 0,\ldots,Q$,$q \neq q_j$, $j \in V_C$;
- $f(q,j) = r_{0j} + lb_j(q)$, $\pi(q,j) = 0$, $q = 0,\ldots,Q$, $j \in V_F$;
- $g(q,j) = \infty$, $\gamma(q,j) = 0$, $q = 0,\ldots,Q$, $j \in V'$.

A q-$*$route is obtained from a q-$*$path ending in i by adding arc $(i,0)$.

5.1.2 Route relaxation ng-$*$route

ng-$*$routes are based on the route relaxations proposed by Baldacci et al. (2011) for the CVRP. Let $N_i \subseteq V'$ be a set of selected nodes for node $i \in V'$ (according to some criterion) such that $N_i \ni i$ and $|N_i| \leq \Gamma$, where Γ is a parameter (e.g., $\Gamma = 5$, $\forall i \in V'$, and N_i contains i and the four nearest nodes to i).

With a forward path $P = (0, i_1, \dots, i_k)$, we associate a set $\Pi(P) \subseteq V'$ defined as:

$$\Pi(P) = \{i_r : i_r \in \bigcap_{s=r+1}^{k} N_{i_s}, r = 1, \dots, k-1\} \cup \{i_k\}. \quad (43)$$

A *forward* ng-$*$path (NG, q, i) is a non-necessarily elementary partial route $P = (0, i_1, \dots, i_{k-1}, i_k = i)$ starting from the depot with a load equal to q, ending at customer i, and such that $NG = \Pi(P)$, and $i \notin \Pi(P')$, where $P' = (0, i_1, \dots, i_{k-1})$. Let $f(NG, q, i)$ be the cost of a least-cost forward ng-$*$path (NG, q, i). The dynamic programming (DP) recursion for computing functions $f(NG, q, i)$ is defined on a state-space graph $\mathscr{H} = (\mathscr{E}, \Psi)$ defined as:

$$\begin{aligned} \mathscr{E} &= \{(NG, q, i) : q_i \leq q \leq Q, \forall NG \subseteq N_i \text{ s.t. } NG \ni i, \forall i \in V\} \\ \Psi &= \{((NG', q', j), (NG, q, i)) : \forall (NG', q', j) \in \Psi^{-1}(NG, q, i),\ \forall (NG, q, i) \in \mathscr{E}\}, \end{aligned} \quad (44)$$

where $\Psi^{-1}(NG, q, i) = \{(NG', q - q_i, j) : \forall NG' \subseteq N_j \text{ s.t. } NG' \ni j \text{ and } NG' \cap N_i = NG \setminus \{i\},\ \forall j \in V \setminus \{i\}\}$, if $i \in V_C$, and $\Psi^{-1}(NG, q, i) = \{(NG', q', j) : 0 \leq q' \leq q - \min_{i \in V_C}\{q_i\}, \forall NG' \subseteq N_j \text{ s.t. } NG' \ni j \text{ and } NG' \cap N_i = NG \setminus \{i\},\ \forall j \in V \setminus \{i\}\}$, if $i \in V_F$.

The DP recursion for computing functions $f(NG, q, i)$, for each state $(NG, q, i) \in \mathscr{E}$ is as follows:

i) $i \in V_F$: $f(NG, q, i) = \min_{(NG', q', j) \in \Psi^{-1}(NG, q, i)}\{f(NG', q', j) + r_{\{j,i\}} + lb_i(q - q')\}$, $\forall (NG, q, i) \in \mathscr{E}$,

ii) $i \in V_C$: $f(NG, q, i) = \min_{(NG', q', j) \in \Psi^{-1}(NG, q, i)}\{f(NG', q', j) + r_{\{j,i\}}\}$, $\forall (NG, q, i) \in \mathscr{E}$,

where functions $lb_i(q)$ are computed as described in Section 5.1.1 and the initialization $f(\{0\}, 0, 0) = 0$ and $f(\{0\}, q, 0) = \infty$, $\forall\ 0 < q \leq Q$ is required. We define a ng-$*$route as a route obtained by adding, to an ng-$*$path (NG, q, i), edge $e = \{0, i\}$; the cost of an ng-$*$route is equal to the cost of ng-$*$path (NG, q, i) plus r_e.

5.1.3 Procedure CG

Let $\overline{\mathscr{R}} \subseteq \mathscr{R}^{>}$ be a subset of routes satisfying a given route relaxation. Moreover, given a route ℓ, we denote with $q(R_\ell) = \sum_{i \in V_C(R)} q_i + \sum_{i \in V_A(R_\ell)} q_i$ its load. Procedure CG works as follows.

Step 1. *Initialization*. Generate a route set $\overline{\mathscr{R}}$ to initialize the master problem which corresponds to LSP, where $\mathscr{R}$ is replaced with $\overline{\mathscr{R}}$. We assume that $\overline{\mathscr{R}}$ contains at least one route containing each customer $i \in V_C$. Set $LCG = 0$ and $iter = 1$.

Step 2. *Find a master dual solution* $\overline{\mathbf{u}}$ *of cost* $\overline{z}$. Initializes $\overline{z} = 0$ and performs $Maxit2$ iterations of the following operations.

(i) Compute a dual solution $\mathbf{u}$ of the master of cost z by means of expressions (30) and (31), where $\mathscr{R}$ is replaced with $\overline{\mathscr{R}}$ and by using the current vector $\boldsymbol{\lambda}$. Let $\tilde{\mathscr{R}}$ be the index set of routes producing ν_i, $i \in V_C$, in expressions (30), and let $\ell(i)$ be the index of the route in $\tilde{\mathscr{R}}$ associated with ν_i, $i \in V_C$. Define a non-necessarily feasible solution $\boldsymbol{\xi}$ of LSP as $\xi_\ell = \sum_{i \in V_C} \overline{a}_{i\ell} \frac{q_i}{q(R_\ell)} \zeta^i_\ell$, $\ell \in \tilde{\mathscr{R}}$, by setting $\zeta^i_{\ell(i)} = 1$ and $\zeta^i_\ell = 0$, $\forall \ell \in \tilde{\mathscr{R}} \setminus \{\ell(i)\}$, $\forall i \in V_C$. If $z > \overline{z}$, update $\overline{z} = z$, $\overline{\boldsymbol{\xi}} = \boldsymbol{\xi}$, $\overline{\mathbf{u}} = \mathbf{u}$.

(i) Update the penalty vectors $\boldsymbol{\lambda}$ as follows. Compute $\alpha_i = \sum_{\ell \in \tilde{\mathscr{R}}} \overline{a}_{i\ell}\xi_\ell$, $i \in V_C$, and $\alpha_i = \sum_{\ell \in \tilde{\mathscr{R}}} a_{i\ell}\xi_\ell$, $i \in V_F$. Then, vector $\boldsymbol{\lambda}$ is modified as follows: $\lambda_i = \lambda_i - \epsilon\gamma(\alpha_i - 1)$, $i \in V_C$, and $\lambda_i = \min\{0, \lambda_i - \epsilon\gamma(\alpha_i - 1)\}$, $i \in V_F$. where ϵ is a positive constant and $\gamma = \frac{0.2\overline{z}}{\sum_{i \in V'}(\alpha_i - 1)^2}$.

Step 3. *Check if* $\overline{\mathbf{u}}$ *is a feasible DSP solution.* Generate the largest subset $\mathscr{N} \subseteq \mathscr{R}^>$ of routes having negative reduced cost with respect to the current dual master solution $\mathbf{u}$ and such that $|\mathscr{N}| \leq \Delta$ (Δ is an a priori defined parameter). If $\mathscr{N} = \emptyset$ and $\overline{z}$ is greater than LCG, then $LCG = \overline{z}$, $\mathbf{u}^* = \overline{\mathbf{u}}$, $\boldsymbol{\xi}^* = \overline{\boldsymbol{\xi}}$ and $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}$; otherwise, $\overline{\mathscr{R}} = \overline{\mathscr{R}} \cup \mathscr{N}$ is updated.

Step 4. *Termination criterion.* Set $iter = iter + 1$. If $iter = Maxit1$, stop.

Computing lower bound LB_1 Lower bound LB_1 corresponds to lower bound LCG computed by procedure CG using q-$*$route relaxation. The initial route set $\overline{\mathscr{R}}$ of the master problem contains a feasible solution generated with the heuristic algorithm described in 6.1. We initialize $\boldsymbol{\lambda} = \mathbf{0}$.

Define the modified routing cost $\overline{r}_{\{i,j\}} = r_{\{i,j\}} - (1/2)(\overline{u}_i + \overline{u}_j)$, $\forall \{i,j\} \in E$ (we assume $\overline{u}_0 = 0$), and the modified assignment cost $\overline{d}_{ij} = d_{ij} - \overline{u}_i$, $\forall (i,j) \in A$, with respect to the current dual solution $\overline{\mathbf{u}}$. The set $\mathscr{N}$ is computed as follows. We compute functions $lb_k(q)$, $f(q,i)$ and $g(q,i)$ using the modified routing and assignment costs $\overline{r}_{\{i,j\}}$ and $\overline{d}_{ij}$ instead $r_{\{i,j\}}$ and d_{ij}. Let $h(i) = \min_{q_i \leq q \leq Q}\{f(q,i) + \overline{r}_{\{0,i\}}\}$, if $\forall i \in V_C$, and $h(i) = \min_{q^{min} \leq q \leq Q}\{f(q,i) + \overline{r}_{\{0,i\}}\}$, $\forall i \in V_F$. The set $\mathscr{N}$ contains any q-$*$route corresponding to $h(i) < 0$, $i \in V'$. Set $\mathbf{u}^1 = \mathbf{u}^*$, $\boldsymbol{\lambda}^1 = \boldsymbol{\lambda}^*$, and $LB_1 = LCG$.

Computing lower bound LB_2 Lower bound LB_2 corresponds to lower bound LCG computed by procedure CG using ng-$*$route relaxation.

We initialize $\boldsymbol{\lambda} = \boldsymbol{\lambda}^1$, define $r^1_{\{i,j\}} = r_{\{i,j\}} - (1/2)(u^1_i + u^1_j)$, $\forall \{i,j\} \in E$ (we assume $u^1_0 = 0$), $d^1_{ij} = d_{ij} - u^1_i$, $\forall (i,j) \in A$, and compute N_i to be the Γ nearest nodes to i according to $r^1_{\{i,j\}}$. We compute functions $f(NG,q,i)$ and $lb_k(q)$ using $r^1_{\{i,j\}}$ and d^1_{ij} instead of $r_{\{i,j\}}$ and d_{ij}, respectively, and the costs $h(i) = min_{(NG,q,i) \in \mathscr{E}}\{f(NG,q,i) + r^1_{\{0,i\}}\}$, of the least cost ng-$*$route visiting i immediately before arriving at the depot. The initial route set $\overline{\mathscr{R}}$ contains the ng-$*$routes corresponding to $h(i) < 0$, $i \in V'$. At each iteration of procedure CG, to generate the set $\mathscr{N}$, we compute functions $f(NG,q,i)$ and $lb_k(q)$ with the modified routing cost $\overline{r}_{\{i,j\}} = r_{\{i,j\}} - (1/2)(\overline{u}_i + \overline{u}_j)$, $\forall \{i,j\} \in E$, and the modified assignment cost $\overline{d}_{ij} = d_{ij} - \overline{u}_i$, $\forall (i,j) \in A$, with respect to the current solution $\overline{\mathbf{u}}$. $\mathscr{N}$ contains every ng-$*$route corresponding to $h(i) = min_{(NG,q,i) \in \mathscr{E}}\{f(NG,q,i) + \overline{r}_{\{0,i\}}\} < 0$, $i \in V'$. Set $LB_2 = LCG$.

5.2 Column-and-cut generation method

In this section, we describe a bounding procedure that computes a lower bound on the VRPTF as the cost of an optimal solution of problem $\overline{LSP}$ obtained from formulation LSP by substituting the route set $\mathscr{R}$ with the set $\mathscr{R}^{>}$ of ng-∗route and by adding valid inequalities derived from a family $\mathscr{F}$ of valid inequalities described for formulation TI.

Any valid inequality $t \in \mathscr{F}$ can be expressed in general form as

$$\sum_{e\in E}\alpha_e^t x_e + \sum_{i\in V'}\beta_i^t y_i + \sum_{(i,j)\in A}\gamma_{ij}^t z_{ij} \geq \omega^t, \tag{45}$$

and can be transformed into the following valid inequality for formulation SP using equations (32)-(35), where $\mathscr{R}$ is substituted by $\mathscr{R}^{>}$:

$$\sum_{\ell\in\mathscr{R}^{>}}(\varphi_\ell^t + \phi_\ell^t + \psi_\ell^t)\xi_\ell \geq \omega^t, \tag{46}$$

where $\varphi_\ell^t = \sum_{e\in E}\alpha_e^t\eta_e^\ell$, $\phi_\ell^t = \sum_{i\in V'}\beta_i^t a_{i\ell}$ and, $\psi_\ell^t = \sum_{(i,j)\in A}\gamma_{ij}^t b_{i\ell}^j$.

The bounding procedure solves problem $\overline{LSP}$ by using column and cut generation. The initial master problem is obtained from the computation of lower bound LB_2 by replacing the route set $\mathscr{R}^{>}$ with the route set $\overline{\mathscr{R}}$ generated by procedure CG during the computation of LB_2. The initial set of valid inequalities $\overline{\mathscr{F}}$ is set to the empty set. At each iteration (say k), the procedure performs the following steps.

1. Solve problem $\overline{LSP}$. Let $\overline{\boldsymbol{\xi}}$ and $(\overline{\mathbf{u}}, \overline{\mathbf{v}})$ be the optimal primal and dual solutions, respectively. Vector $\overline{\mathbf{u}}$ is given by $\overline{\mathbf{u}} = \{\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{|V_C|}, \overline{u}_{|V_C|+1}, \ldots, \overline{u}_{|V'|}\}$, where $u_0 = 0$ and $\overline{u}_1, \ldots, \overline{u}_{|V_C|}$ are associated with constraints (27), and $\overline{u}_{|V_C|+1}, \ldots, \overline{u}_{|V'|}$, with constraints (28). Vector $\overline{v} = \{\overline{v}_1, \ldots, \overline{v}_{|\overline{\mathscr{F}}|}\}$ is associated with the family of valid inequalities $\overline{\mathscr{F}}$.

2. Generate the largest subset $\mathscr{N} \subseteq \mathscr{R}^{>}$ of ng-∗route having negative reduced cost with respect to the current dual master solution $(\overline{\mathbf{u}}, \overline{\mathbf{v}})$ and such that $|\mathscr{N}| \leq \Delta$ (Δ is an a priori defined parameter). If $\mathscr{N} = \emptyset$, the procedure terminates; otherwise a new iteration is made. At iteration $k+1$, the procedure solves a new master problem $\overline{LSP}$ by replacing $\overline{\mathscr{R}}$ with $\overline{\mathscr{R}} \cup \mathscr{N}$ and the valid inequalities of $\mathscr{F}$ violated by the $\overline{LSP}$ solution $\overline{\xi}$ achieved by iteration k.

3. Given the solution vector $\overline{\boldsymbol{\xi}}$, compute the corresponding solution vector $(\overline{\mathbf{x}}, \overline{\mathbf{z}}, \overline{\mathbf{w}})$ by means of equations (32)-(35) where $\mathscr{R}$ is substituted by $\overline{\mathscr{R}}$. Solve the separation problems associated with the set of valid inequalities $\mathscr{F}$ (see below) and add, if any, violated inequalities to set $\overline{\mathscr{F}}$.

It can be easily shown that the complexity of the pricing algorithm solved at Step 2 of the above procedure is not sensitive to the addition of the valid inequalities in $\overline{\mathscr{F}}$, since the values of the corresponding dual variables can be translated into subproblem costs. Indeed, at each iteration of the procedure, to generate the set $\mathscr{N}$, we compute the ng-∗route functions $f(NG, q, i)$ and $lb_k(q)$ with the modified routing cost $\overline{r}_{\{i,j\}} = r_{\{i,j\}} - (1/2)(\overline{u}_i + \sum_{t\in\overline{\mathscr{F}}}\beta_i^t\overline{v}_t) - (1/2)(\overline{u}_j + \sum_{t\in\overline{\mathscr{F}}}\beta_j^t\overline{v}_t) - \sum_{t\in\overline{\mathscr{F}}}\alpha_{\{i,j\}}^t\overline{v}_t$, $\forall\{i,j\} \in E$, and the modified assignment cost $\overline{d}_{ij} = d_{ij} - \overline{u}_i - \sum_{t\in\overline{\mathscr{F}}}\gamma_{ij}^t\overline{v}_t$, $\forall(i,j) \in A$, with respect to the current dual solution $(\overline{\mathbf{u}}, \overline{\mathbf{v}})$. $\mathscr{N}$ contains every ng-∗route corresponding to $h(i) < 0$, $i \in V'$.

We conducted preliminary experiments to identify a good separation strategy to be used at Step 3. As a result of our experimentation, we decided to use the following inequalities to define the family set $\mathscr{F}$: CI, MI, RCI, RCII-a, RCII-b, RCII-c, and RCII-d inequalities. For a given solution $(\overline{\mathbf{x}}, \overline{\mathbf{z}}, \overline{\mathbf{w}})$, we identified (as far as possible) violated inequalities of above seven types by applying the corresponding separation procedures as described below.

5.2.1 Separation procedures

The separation problems of CI, RCII-a and RCII-c inequalities can be reduced to max-flow/min-cut problems using a standard construction, and therefore solved in polynomial time; we omit the details for sake of brevity (see Baldacci et al. (2007)). Concerning MI inequalities, the following theorem holds.

Theorem 4 *Let* (x, z, y) *be a solution of the LP-relaxation of formulation* TI *and assume that* $q_i \leq Q$, $\forall i \in V_C$, *and that* $x_e = 0$, $e = \{i, j\} \in E \setminus \{\{0, h\} : h \in V'\}$, *if* $q_i + q_j > Q$. *The separation problem for MI inequalities* (10) *is solvable in polynomial time.*

Proof. The proof is provided in the e-companion to this paper. □

RCI, RCII-b and RCII-d inequalities are separated using a heuristic separation procedure. The procedure is a Multistart Local Search that, at each iteration, generates a starting point and evolves it through a Local Search procedure. We start by generating a set $\mathscr{S}$ of $10(n-1)$ subsets of V' as follows. For the RCI inequalities the first $|V_C|$ sets of $\mathscr{S}$ are obtained by inserting in each set, for $i = 1, \ldots, |V_C|$, the nodes in F_i. The remaining sets are generated by first computing a random number m drawn from a uniform distribution in $[1, \ldots, n-1]$, and then by randomly selecting m different nodes of V', again using a uniform distribution. For the RCII-b and RCII-c inequalities all the sets are randomly generated as above. Each set $S \in \mathscr{S}$ is then iteratively expanded by adding one node at each iteration until $S = V'$. For a given set S, let $\theta(S)$ denote the difference between the left-hand side and the right-hand side value of the considered inequality (i.e., the inequality can be rewritten as $\theta(S) \geq 0$ and the separation problem corresponds to compute $\arg\min_{S \subseteq V'}\{\theta(S)\}$). Each set S is expanded by choosing the node $i \in V' \setminus S$ such that $\theta(S \cup \{i\})$ is minimized

6 Solving the VRPTF to Optimality

In this section, we describe the method implemented for solving the VRPTF to optimality. We start by describing two heuristic algorithms that compute primal bounds used to initialize the exact method. The exact method is a branch-and-cut-and-price (BCP) solution method based on the SCIP (see Achterberg 2009) BCP solution framework.

6.1 Heuristic algorithms

Primal bounds for the VRPTF are computed by means of two different types of heuristic algorithms: a *constructive heuristic* and a *Lagrangean heuristic.*

The basis of the constructive algorithm is a heuristic to solve the CVRP. Given an instance of VRPTF, we define a complete graph $\overline{G} = (\overline{V}, \overline{E})$ where the node set $\overline{V} = \{0\} \cup V_C$ contains the depot and the customer nodes. Each edge $e \in \overline{E}$ has a cost given by r_e. Each customer $i \in V_C$ has a demand equal to q_i and the capacity of the vehicles is set to Q. Roughly speaking, we solve a problem obtained from VRPTF by disregarding the facility nodes (set V_F) and the connection arcs (set A). The CVRP instance is solved through an iterative multistart procedure based on a cluster-first, route-second heuristic procedure. Each iteration consists of three phases: (i) determine a partition of the customers into a number of subsets each one satisfying the capacity constraint; (ii) for each set, find the route of a single vehicle that serves all the customers in the set (i.e. we solve an instance of a Traveling Salesman Problem (TSP)); (iii) locally optimize the solution obtained at step (ii). The CVRP solution so far obtained, is then locally optimized by iteratively applying two post-optimization procedures specifically devised for the VRPTF.

The Lagrangean heuristic is based on procedure CG described in Section 5.1.3. Procedure CG is interwoven with an algorithm that produces a feasible VRPTF solution using the route set $\tilde{\mathscr{R}}$ (see Step 2 of procedure CG). The route set $\tilde{\mathscr{R}}$ is first modified to contain only customers visited at most once. Then, unrouted customers are inserted in order to obtain a feasible solution. The solution obtained is further optimized by applying the same post-optimization procedures used by the constructive algorithm.

A step-by-step description of the heuristics are given in the e-companion to this paper.

6.2 Details of the BCP method

The lower bound at the root node of the enumeration tree is first computed by using the bounding procedure described in Section 5, then by using the column-and-cut generation method described in Section 5.2. The master problem at a generic node except the root node is initialized with the set of valid inequalities $\overline{\mathscr{F}}$ and the set of routes $\overline{\mathscr{R}}$ of the parent node, where set $\overline{\mathscr{R}}$ is further modified by extracting the largest set of routes satisfying the branching conditions.

To choose a node-selection rule, we first performed some preliminary experiments with different rules and, based on these results, we decided to adopt the *best-first strategy* for all the computations of Section 7. We did not implement primal heuristics but the algorithm was initialized with the best primal solution found by the two heuristic algorithms described in the previous section that are executed at the root node. We used the default branching scheme of the SCIP framework, namely the *hybrid branching* scheme (see Achterberg and Berthold 2009), that combines ideas from pseudocost branching (Benichou et al. 1971) and strong branching (Applegate et al. 2007).

7 Computational Results

This section reports on the computational results of the exact method described in this paper and analyses the effectiveness of the dual ascent heuristics and of the different types of inequalities on the bounding procedure procedure described in Section 5.

The algorithms were coded in C++ and linked with the SCIP 3.1.1 BCP solution framework (see Achterberg 2009) using the IBM Cplex 12.6.1 linear programming solver (see IBM CPLEX 2014). The experiments were performed on an Intel Core 2 Duo at 2.66 GHz personal computer equipped with 4 Gb of RAM.

The exact method has been tested on real-world instances and on instances derived from LRP instances already proposed in the literature, used to further evaluate the performance of our algorithms. The same instances have been also used to generate 2E-CVRP instances. The following sections 7.1 and 7.2 briefly describe the real-world and LRP based instances, respectively, and report on the results obtained by the different algorithms. The complete details of the instances are provided in the e-companion to this paper.

Based on the results of preliminary experiments to identify good parameter settings for our method, we decided to use the following settings for our bounding procedure (see Section 5):

- in computing lower bound LB_1: $Maxit1 = 50$, $Maxit2 = 50$, $\epsilon = 1.5$ and $\Delta = 50$;
- in computing lower bound LB_2: $\Gamma = 12$, $Maxit1 = 100$, $Maxit2 = 50$, $\epsilon = 2.0$ and $\Delta = 50$;
- in the column-and-cut-generation method: $\Delta = 100$ at the root node of the BCP whereas $\Delta = 50$ for the remaining nodes.

7.1 Results on real-world instances

The data of this set of instances were provided by a major Italian transportation company that distributes non-perishable products over the whole Italian peninsula. The company operates through three main distribution areas (*North*, *Centre* and *South*) using three main central depots located in the provinces of Milan, Rome and Naples.

The three distribution areas operate independently in the corresponding areas to serve customer orders using an existing set of intermediate facilities. The customer orders are placed into *Euro*-pallet and distributed either to the final customers or the intermediate facilities by means of a fleet of identical capacitated vehicles which are stationed at the different central depots and whose capacity is expressed in terms of pallets. All the facilities are owned by third-party contractors, that are in charge of delivering to the final customers the orders consolidated at the facilities.

The company was interested in analyzing different distribution scenarios associated with the three distribution areas. A total number of 18 instances were provided by the company, six instances per each area or depot. The following naming convention was adopted to identify the different instances. The instance name is a string **area_a**×**b_**Q**c**, where **area** represents the area (i.e., *North*, *Centre*, *South*), **a** represents the number of customers, **b** corresponds to the number of facilities, and **c** is the vehicle capacity.

In Table 1, we report the results obtained by the heuristic algorithms, the bounding procedure and the BCP method. The columns of the table report the instance name (*Name*), the cost of the best solution found by the heuristics and BCP algorithms (z^*), the percentage deviation of the upper bound computed by the constructive heuristic ($\%UB_1$), the percentage deviation of the upper bound computed by the lagrangean heuristic ($\%UB_2$), the percentage deviation of lower bound LB_1 ($\%LB_1$), the percentage deviation of lower bound LB_2 ($\%LB_2$), the total computing time of lower bounds LB_1 and LB_2 that also includes the time spent for computing UB_2 (t_{DA}), the percentage deviation of the lower bound LB computed at the root-node of the BCP algorithm and the corresponding computing time ($\%LB$, t_{LB}), the cardinality of the sets $\overline{\mathscr{F}}$ and $\overline{\mathscr{R}}$ associated with lower bound LB (#*cuts* and #*cols*), the total number of nodes of the exact algorithms (#N), the percentage deviation of the best lower bound achieved by the exact method (%*Opt*), and the total computing time in seconds spent by the exact method (t_{TOT}), that also include the time spent for computing upper bound UB_1. The percentage deviation of value x is computed as $100 \times x/z^*$.

Table 1: Results on real-world instances

Name	z^*	$\#r$	$\#f$	$\#c$	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_C$	t_C	$\%LB$	t_{LB}	$\#cuts$	$\#cols$	#N	$\%Opt$	t_{TOT}
north-68x7-Q24	8890.3	23	6	27	106.8	102.2	97.8	97.9	51.7	95.5	38.7	99.1	2.3	18	1093	1930	100.0	565
north-68x7-Q34	9748.1	17	5	25	105.0	100.7	96.3	96.3	36.5	93.8	28.1	97.1	5.3	496	1535	4126	100.0	5426
north-103x13-Q24	14251.1	34	13	46	105.8	102.2	98.1	98.1	91.6	94.3	141.8	98.9	4.5	23	1260	9888	99.5	7466
north-103x13-Q34	15613.3	27	12	61	107.2	102.1	98.4	98.5	164.8	95.7	204.2	99.1	10.7	489	2786	6641	99.6	7497
north-142x18-Q24	17876.4	49	16	65	110.7	102.8	98.7	99.0	201.4	93.5	396.8	99.3	17.1	3	3490	3382	100.0	3172
north-142x18-Q34	19623.3	39	16	74	118.0	103.3	97.5	97.8	249.0	92.7	515.4	98.3	19.2	2	3984	1583	98.7	7765
centre-74x6-Q24	12213.8	24	6	21	106.6	100.7	99.6	99.9	47.5	95.7	53.0	99.9	2.6	494	1287	3	100.0	120
centre-74x6-Q34	12930.4	19	5	28	104.6	100.9	99.1	99.6	60.3	96.0	57.1	99.7	4.2	188	1628	92	100.0	205
centre-113x9-Q24	19612.1	38	8	41	105.4	102.0	99.2	99.3	260.9	94.8	319.5	99.5	11.3	4	2464	4954	100.0	2578
centre-113x9-Q34	21877.0	31	8	41	106.7	101.4	97.2	97.6	232.4	94.1	325.4	97.7	8.1	2	4283	1639	98.1	7595
centre-164x12-Q24	27390.2	56	12	46	108.3	102.3	99.0	99.1	656.5	93.2	915.2	99.3	41.2	11	4411	3863	99.4	8452
centre-164x12-Q34	29853.5	45	11	65	113.9	102.6	99.0	99.1	910.3	94.7	1493.3	99.4	32.9	14	6055	2087	99.5	8565
south-54x4-Q24	10987.6	19	4	16	102.1	100.1	97.7	98.2	33.2	96.4	21.5	98.8	0.8	2	678	4586	100.0	1328
south-54x4-Q34	12597.7	26	4	13	106.9	100.5	94.4	94.4	39.5	94.6	26.0	95.6	1.9	33	1200	1030	96.2	7255
south-85x7-Q24	16553.7	29	7	22	107.1	100.0	97.0	96.9	116.3	93.8	101.3	97.6	5.0	8	2220	2476	97.7	7421
south-85x7-Q34	18100.4	22	7	31	105.7	102.8	98.5	98.9	200.9	96.6	126.0	99.4	6.6	389	3013	1729	100.0	5470
south-115x9-Q24	20497.5	39	8	38	104.9	102.3	98.5	98.2	424.5	94.3	283.0	99.2	10.3	3	2726	9148	99.7	7878
south-115x9-Q34	21963.2	33	9	52	107.0	101.2	97.4	97.6	342.7	93.8	187.5	98.5	10.5	2	4685	1819	98.9	7737
					107.4	101.7	98.0	98.1	228.9	94.6	290.8	98.7	10.8					2358

In order to evaluate the quality of the different lower bounds, we also computed, for each instance, the value of the lower bound obtained by solving the LP-relaxation of formulation TI strengthened with the different valid inequalities (using the separation strategy described in Section 5.2). In the table, column $\%LB_C$ reports the percentage deviation of the final lower bound obtained whereas column t_C displays the corresponding computing time.

For each instance, Table 1 also reports the following details about the best solution found: the number of routes in the solution ($\#r$), the number of facilities visited ($\#f$) and the number of customers assigned to a facility ($\#c$).

For these set of instances, a time limit of 7,200 seconds was imposed to the SCIP framework.

The last row of the table reports averages computed over the different columns. The average reported under column t_{TOT} is computed over the instances solved to optimality within the imposed time limit. If a value of 100.0 is reported for column $\%Opt$, then the algorithm terminated with an optimal solution.

Table 1 shows that 8 out of 18 instances were solved to optimality and that the final lower bound LB is on average quite tight, being equal to 98.7%. The largest instance solved to optimality involves 142 customers and 18 facilities. On these set of instances, lower bounds LB_1 and LB_2 have the same quality and are on average superior to lower bound LB_{CP}, thus showing the effectiveness of our q-$*$route and ng-$*$route relaxations. Moreover, the different valid inequalities can substantially increases the lower bound, as shown by the improvements on instances north-68x7-Q24 and south-54x4-Q34.

The table shows that upper bound UB_2 is always better than upper bound UB_1 and that the BCP algorithm can further improve the upper bounds in almost all instances, thus producing high quality primal solutions also whenever the algorithm terminates without proving the optimality of the solution found.

It is worth mentioning that the time spent for computing upper bound UB_1 is on average equal to 187.4 seconds and that the time spent by the procedure used to compute upper bound UB_2 (called during the computation of lower bound LB_2) is on average equal to 226.8 seconds. Therefore, both UB_1 and UB_2 can be computed efficiently in practice.

7.2 Results on LRP based instances

This set of instances was derived from 75 LRP instances used in Baldacci et al. (2011) and Contardo et al. (2013) for solving the LRP and proposed by different authors. We derived two classes of test instances (A and B) having the same topology of the underlying graph, but with different cost structures.

We generated a total number of 150 instances, 75 instances per class. The dimensions of the instances vary from very small instances with 12 customers and two facilities up to large instances with 150 customers and 20 facilities. The instance name is a string **name<a×b>**, where **name** represents the instance name, **a** represents the number of customers and **b** corresponds to the number of facilities.

For sake of presentation, the instances were grouped into the following three groups accordingly to the original LRP source:

i) Akca et al. (2009): 12 instances involving 5 facilities, and 30 or 40 customers;

Table 2: Summary results on Class A instances

	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_{CP}$	t_{CP}	$\%LB$	t_{LB}	$\#Opt$	t_{TOT}
Akca et al. (2009)	100.3	100.6	94.3	96.8	9.9	96.1	4.6	98.5	2.1	10/12	145.3
Prins et al. (2004)	100.2	100.4	93.7	96.0	48.6	94.0	78.6	97.8	14.7	10/24	221.9
Different authors	100.2	101.0	91.9	94.3	297.5	93.7	339.8	96.6	121.3	10/39	213.0

Table 3: Summary results on Class B instances

	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_{CP}$	t_{CP}	$\%LB$	t_{LB}	$\#Opt$	t_{TOT}
Akca et al. (2009)	102.6	101.3	94.6	96.9	5.2	95.6	4.1	98.1	3.7	9/12	274.2
Prins et al. (2004)	101.4	101.0	94.2	95.9	52.8	93.0	76.1	97.2	14.2	7/24	184.0
Different authors	101.1	102.5	92.4	94.2	187.6	93.4	324.1	96.1	141.2	8/39	284.3

ii) Prins et al. (2004): 24 instances involving 20, 50, and 100 customers, 5 or 10 facilities;

iii) Different authors: 39 instances, involving up to 150 customers and 20 facilities.

For this set of instances, a time limit of 3,600 seconds was imposed to the SCIP framework.

Tables 2 and 3 summarize the results obtained on both classes A and B. In the tables, column $\#Opt$ reports for each group of instances the total number of instances solved to optimality within the imposed time limit.

The meaning of the remaining columns is the same described in the previous section, but in the tables their values are relative to averages computed over the instances composing the three groups. The values reported under column t_{TOT} are computed over the instances solved to optimality within the imposed time limit.

Tables 2 and 3 show that 30 and 24 out of 75 instances were solved to optimality within the imposed time limit for classes A and B, respectively.

For these instances, lower bound LB_2 is on average superior with respect lower bound LB_1. As the feasible solutions associated with these instances are characterized (on average) by a larger number of customers per route, the ng-$*$route relaxation performs in practice better than q-$*$route relaxation. Also for these instances, the different valid inequalities can substantially increase the final lower bound (see column $\%LB$). Instances of Class B are more difficult with respect to the corresponding instances of class A. This is due to the different cost structure of class B instances and it is testified by the worse quality of lower bounds LB_{CP} and of the final lower bound LB. Nonetheless, lower bounds LB_1 and LB_2 show the same quality of class A instances.

Concerning the upper bounds, the tables show that both the two upper bounding procedures can compute good quality solutions. The average computing time of upper bound UB_1 (UB_2) is equal to 70.8 and 72.9 seconds (148.0 and 89.5 seconds) for classes A and B, respectively. Therefore, the computation of LB_2 requires a higher computing time with respect to the real-world instances and this is due to the larger vehicle capacity that characterizes most of the instances in classes A and B.

The detailed results reported in the e-companion to this paper show that instances with up to 100

Table 4: Effectiveness of the dual ascent heuristics

					$t_{LB_{SP}}$				t_{LB_1}	
		$\%LB_1$	$\%LB_2$	$\%LB_{SP}$	(a)	(b)	(c)	(d)	(e)	(f)
A	Akca et al. (2009)	94.3	96.8	96.9	11.1	6.7	4.9	2.3	0.9	0.7
	Prins et al. (2004)	93.7	96.0	96.1	26.3	15.6	12.1	5.3	0.5	0.3
	Different authors	91.9	94.3	94.7	263.0	171.2	108.0	53.5	21.4	17.6
B	Akca et al. (2009)	94.6	96.9	96.9	13.0	7.9	5.8	2.7	0.9	0.7
	Prins et al. (2004)	94.2	95.9	96.0	26.4	17.4	13.2	5.8	0.5	0.4
	Different authors	92.4	94.2	94.8	248.3	168.6	116.5	56.9	21.7	18.9
Real-word		98.0	98.1	98.6	18.9	5.0	8.3	4.2	0.3	0.2
		93.6	95.5	95.8	129.9	85.2	57.4	28.0	10.3	8.7

(a) without lower bounds LB_1 and LB_2
(b) with lower bound LB_1
(c) with lower bound LB_2
(d) with lower bounds LB_1 and LB_2
(e) route set $\overline{\mathscr{R}}$ initialized with single-customer route
(f) route set $\overline{\mathscr{R}}$ initialized with the solution provided by the constructive heuristic

customers and 10 facilities were solved to optimality.

7.3 Effectiveness of the dual ascent heuristics and valid inequalities

Table 4 reports an analysis of the effectiveness of the dual ascent heuristics when used to initialize the master problem of problem $\overline{LSP}$ (see Section 5.2). In order to assess the quality of lower bounds LB_1 and LB_2, we solved problem $\overline{LSP}$ without adding valid inequalities, i.e., we computed the optimal solution cost LB_{SP} of formulation LSP and the LP-relaxation of formulation SP with ng-$*$route. In addition, the Lagrangean heuristic has been disabled during the computation of LB_1 and LB_2.

The table reports the average percentage deviations of lower bounds LB_1, LB_2, and LB_{SP} under columns $\%LB_1$, $\%LB_2$ and $\%LB_{SP}$, respectively. The table then reports, under heading $t_{LB_{SP}}$, the average total computing times spent in computing lower bound LB_{SP} under the following options: (a) without computing lower bounds LB_1 and LB_2 (b) by computing lower bound LB_1 (c) by computing lower bound LB_2, and (d) by computing both lower bounds LB_1 and LB_2. In case (a), the master problem of LSP is initialized with single-customer routes whereas in case (b), the master problem is initialized using the dual solution provided by lower bound LB_1, that is used to generate an initial set of ng-$*$route. In cases (c) and (d), the master problem is initialized with the route set generated by procedure CG during the computation of LB_2 (as described in 5.2). Moreover, in case (c) the master problem associated with the computation of LB_2, is initialized as for LB_1, i.e., using the solution provided by the constructive heuristic described in Section 6.1.

All values in the table are relative to averages computed over the instances composing the three groups of classes A and B, and over the real-world instances. The last row of the table reports averages computed over all instances.

The table shows that the bounding procedure based on the use of both lower bounds LB_1 and LB_2 (case (d)) is about five times faster than the standard column generation method (case (a)). Generally speaking, standard column generation methods are time-consuming as the LP-relaxation of the master problem is usually highly degenerate and degeneracy implies alternative optimal dual solutions. Consequently, the generation of new columns and their associated variables may not change the value of the objective function of the master problem, the master problem may become

Table 5: Effectiveness of the different type of inequalities on column-and-cut generation procedure

		no cuts		+ CI + MI + RCI			+ RCII-a + RCII-b			+ RCII-c + RCII-d		
		$\%LB$	t_{LB}	$\%LB$	t_{LB}	$\#cuts$	$\%LB$	t_{LB}	$\#cuts$	$\%LB$	t_{LB}	$\#cuts$
A	Akca et al. (2009)	96.9	2.3	97.9	3.1	7.3	98.5	4.0	200.6	98.5	4.0	169.9
	Prins et al. (2004)	96.1	5.3	97.1	10.0	8.3	97.8	17.2	521.5	97.8	18.2	539.8
	Different authors	94.7	53.5	95.7	92.1	23.8	96.5	139.3	1083.2	96.6	164.4	1181.3
B	Akca et al. (2009)	96.9	2.7	97.4	3.6	13.3	98.0	5.5	655.6	98.1	5.6	763.0
	Prins et al. (2004)	96.0	5.8	96.3	10.7	14.6	97.1	16.2	623.8	97.2	17.7	737.0
	Different authors	94.8	56.9	95.6	109.0	61.4	96.0	173.3	1385.3	96.1	184.3	1566.1
Real-word		98.6	4.2	98.7	6.6	4.1	98.7	6.9	106.0	98.7	12.9	121.1
		95.8	28.0	96.5	50.8	24.9	97.1	78.8	809.2	97.2	88.1	905.2

large, and the overall method may become slow computationally. In case (d), the bounding procedure starts from a near-optimal dual solution of the LP-relaxation of SP with ng-$*$route provided by lower bound LB_2, as shown by the percentage deviations of lower bounds LB_2 and LB_{SP}. This allows us to generate an initial master problem containing the routes having a very small reduced cost that are likely to be in the optimal LSP solution.

The analysis of cases (b) and (c) shows that it is also computationally convenient to compute LB_1 or LB_2. In particular, computing LB_1 before the computation of LB_2 speedup the computation of LB_2 as procedure CG used to compute LB_2 takes advantage from the master initialization provided by the dual solution corresponding to LB_1.

Table 4 also reports the computational results obtained when calculating the lower bound LB_1 under the following two ways of initializing the corresponding master problem: (i) by using the heuristic solution provided by the constructive heuristic (case (e)) (ii) by using single-customer routes (case (f)). The table shows that on average, the difference is slightly marginal. Nevertheless, as in our implementation the constructive heuristic is executed before computing LB_1, it is worthwhile to initialize the master of LB_1 with the solution found by the heuristic.

Table 5 analyses the impact of the valid inequalities on the column-and-cut bounding procedure described in Section 5.2 at the root node of the BCP method.

The table reports average percentage deviations of the lower bounds obtained by the bounding procedure under the following cases: (i) without adding valid inequalities (under column heading "no cuts") (ii) by adding CI, MI and RCI inequalities ("+ CI + MI + RCI") (iii) by adding CI, MI, RCI, RCII-a, and RCII-b inequalities ("+ RCII-a + RCII-b"), and (iv) by adding CI, MI, RCI, RCII-a, RCII-b, RCII-c, RCII-d inequalities ("+ RCII-c + RCII-d"). The last case corresponds to the final procedure we adopted in our computational results and, as mentioned in Section 5.2, the sequence of separation procedures was defined after conducting preliminary computational experiments performed to identify a good separation strategy.

For each group, the table reports the average percentage deviations of the lower bounds obtained and the corresponding average computing times ($\%LB$, t_{LB}), and the average cardinalities of the sets $\overline{\mathscr{F}}$ associated with the lower bound computation ($\#cuts$). As for Table 4, the Lagrangean heuristic has been disabled during the computation of LB_1 and LB_2. In addition, the time t_{LB} also includes the time spent for computing LB_1 and LB_2.

As for Table 4, all values in the table are relative to averages computed over the instances composing the three groups of classes A and B, and over the real-world instances. The last row of the table reports averages computed over all instances.

The table shows that the average percentage gaps left by considering in turn the different three groups of valid inequalities are equal to 3.5, 2.9 and 2.8, respectively. With respect to the "no cuts" case, a final gap reduction of about 1.4% has been achieved. The contribution given by inequalities RCII-c and RCII-d is on average equal to 0.1% as shown by the table. During preliminary computational experiments, we observed that their addition generally results in separating additional RCI and RCII-b inequalities, which separation procedures are heuristics.

8 Conclusions

In this paper, we considered a vehicle routing problem with transhipment facilities, called the Vehicle Routing Problem with Transhipment Facilities (VRPTF), that was motivated by a real-world application of interest to an Italian company operating in the production and distribution of non-perishable products. The VRPTF consists of selecting transhipment facilities, allocating customers to these facilities and designing vehicle routes emanating from a central depot to minimize the total distribution cost. A feature of the problem is that a customer can be either served on a vehicle route emanating from the central depot or through an intermediate facility, where the demand is first delivered by a vehicle route, and then it is successively delivered to the final customer.

We proposed two integer programming formulations for the VRPTF, a two-index formulation (TI) and a set-partitioning based formulation (SP). The formulations were used to derive a bounding method based on two dual ascent heuristics and a column-and-cut generation procedure. In particular, we proposed valid inequalities to strengthen the linear relaxations of the two formulations and two different route relaxations, called q-$*$route and ng-$*$path, that have the advantage that the pricing subproblem associated with the linear relaxation of formulation SP can be efficiently solved (by dynamic programming).

All our findings have been used to develop branch-and-cut-and-price algorithm that has been tested on a large family of instances, including both real-world instances and instances derived from the literature.

The implementation solved to optimality different instances from our real-world instances involving up to 142 customers and 18 facilities. The implementation was also tested on literature-based instances to better evaluate the limits of the algorithms, and the new approaches can find optimal solutions on some difficult instances with up to 100 customers and 10 facilities.

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9 Proofs of statements

Proposition 1 *There are no dominance relations between inequalities RCII-a and RCII-b.*

Proof. Consider a VRPTF instance with $|V_C| = 6$, $|V_F| = 1$, with $V_C = \{1, 2, 3, 4, 5, 6\}$ and $V_F = \{7\}$. In addition, let $Q = 10$ and $q_1 = 4$, $q_2 = 3$, $q_3 = 5$, $q_4 = q_5 = q_6 = 2$. First select a set $S = \{1, 2, 4, 7\}$ and such that the nodes $\{1, 2, 4, 7\}$ are visited on a route while customers $\{3, 5, 6\}$ are assigned to facility node 7 of S. The right-hand side of inequality (15) becomes $\lceil 18/10 \rceil - 0 = 2$, while the right-hand side of (17) has value $\lceil 9/10 \rceil - 0 = 1$ and (15) is stronger than (17). Now consider a set $S = \{1, 2, 3, 4, 5\}$ containing five customers all visited on a route. Customer 6 is associated with a node in $V' \setminus S$. The right-hand side of (15) takes value $\lceil 18/10 \rceil - 1/4 = 7/4$, while the right-hand side of (17) becomes $\lceil 16/10 \rceil - 0 = 2$ and the second inequality is stronger than the first one. Moreover, note that the FrCC inequalities (4) are dominated by (15), for the first example, and by both (15) and (17) for the second one. □

Lemma 3 (Lemma 2.) *Let $o \in \mathbb{R}$ with $\hat{o} > 0$ and $T = \{m \in \mathbb{R}, n \in \mathbb{Z} : m + n \geq o, m \geq 0\}$. The inequality*

$$m + \hat{o}n \geq \hat{o}\lceil o \rceil \tag{18}$$

is valid for T.

Proof. We have two cases:

(i) $n \geq \lceil o \rceil$. As $m \geq 0$, we have $\frac{m}{\hat{o}} \geq 0$, hence $\frac{m}{\hat{o}} + n \geq \lceil o \rceil$;

(ii) $n \leq \lfloor o \rfloor$. As $0 < \hat{o} < 1$ we have that

$$\lfloor o \rfloor - n \geq \hat{o}(\lfloor o \rfloor - n). \tag{47}$$

Since $o = \lfloor o \rfloor + \hat{o}$ and using inequality (47), inequality $m + n \geq o$ can be rewritten as:

$$m \geq \hat{o} + \hat{o}(\lfloor o \rfloor - n). \tag{48}$$

The right-hand side of inequality (48), can be rewritten as:

$$\hat{o}(1 + \lfloor o \rfloor) - \hat{o}n = \hat{o}\lceil o \rceil - \hat{o}n \tag{49}$$

thus obtaining $m \geq \hat{o}\lceil o \rceil - \hat{o}n$. □

Theorem 5 *Let $\alpha_e \geq 0$, $\forall e \in E$, $\beta_i \geq 0$, $\forall i \in V'$ and $\gamma_{ij} \geq 0$, $\forall (i, j) \in A$ and consider the following inequality valid for formulation TI:*

$$\sum_{e \in E} \alpha_e x_e + \sum_{i \in V'} \beta_i y_i + \sum_{(i,j) \in A} \gamma_{ij} z_{ij} \geq o \tag{19}$$

where $o \in \mathbb{R}$ and $\hat{o} > 0$. Then the following inequality:

$$\sum_{e \in E} \varphi^o(\alpha_e)x_e + \sum_{i \in V'} \varphi^o(\beta_i)y_i + \sum_{(i,j) \in A} \varphi^o(\gamma_{ij})z_{ij} \geq \lceil o \rceil \tag{20}$$

where $\varphi^o(m) = \lfloor m \rfloor + \min\left\{\frac{\hat{m}}{\hat{o}}, 1\right\}$, $m \in \mathbb{R}$, $o \in \mathbb{R}$, $\hat{o} > 0$, is also a valid inequality for formulation TI.

Proof. Let $E^1 \subseteq E$, $E^2 = E \setminus E^1$, $V^1 \subseteq V'$, $V^2 = V' \setminus V^1$ and $A^1 \subseteq A$, $A^2 = A \setminus A^1$. Starting from inequality (19) we can round up the coefficients in E^2, V^2 and A^2 to obtain:

$$\sum_{e \in E^1} \alpha_e x_e + \sum_{i \in V^1} \beta_i y_i + \sum_{(i,j) \in A^1} \gamma_{ij} z_{ij} + \sum_{e \in E^2} \lceil \alpha_e \rceil x_e + \sum_{i \in V^2} \lceil \beta_i \rceil y_i + \sum_{(i,j) \in A^2} \lceil \gamma_{ij} \rceil z_{ij} \geq o. \tag{50}$$

Writing $\alpha_e = \lfloor \alpha_e \rfloor + \hat{\alpha_e}$, $\forall e \in E^1$, $\beta_i = \lfloor \beta_i \rfloor + \hat{\beta_i}$, $\forall i \in V^1$, and $\gamma_{ij} = \lfloor \gamma_{ij} \rfloor + \hat{\gamma_{ij}}$, $\forall (i,j) \in A^1$ and re-arranging terms, we get:

$$\begin{aligned}&(\sum_{e \in E^1} \hat{\alpha_e} x_e + \sum_{i \in V^1} \hat{\beta_i} y_i + \sum_{(i,j) \in A^1} \hat{\gamma_{ij}} z_{ij}) + \\ &(\sum_{e \in E^1} \lfloor \alpha_e \rfloor x_e + \sum_{e \in E^2} \lceil \alpha_e \rceil x_e + \sum_{i \in V^1} \lfloor \beta_i \rfloor y_i + \sum_{i \in V^2} \lceil \beta_i \rceil y_i + \sum_{(i,j) \in A^1} \lfloor \gamma_{ij} \rfloor z_{ij} + \sum_{(i,j) \in A^2} \lceil \gamma_{ij} \rceil z_{ij}) \geq o.\end{aligned} \tag{51}$$

The first part of inequality (51) is non-negative, and the second part is integral for all x, w and z integral. Applying Lemma 2 we get:

$$\begin{aligned}&\frac{1}{\hat{o}}(\sum_{e \in E^1} \hat{\alpha_e} x_e + \sum_{i \in V^1} \hat{\beta_i} y_i + \sum_{(i,j) \in A^1} \hat{\gamma_{ij}} z_{ij}) + \\ &(\sum_{e \in E^1} \lfloor \alpha_e \rfloor x_e + \sum_{e \in E^2} \lceil \alpha_e \rceil x_e + \sum_{i \in V^1} \lfloor \beta_i \rfloor y_i + \\ &\sum_{i \in V^2} \lceil \beta_i \rceil y_i + \sum_{(i,j) \in A^1} \lfloor \gamma_{ij} \rfloor z_{ij} + \sum_{(i,j) \in A^2} \lceil \gamma_{ij} \rceil z_{ij}) \geq \lceil o \rceil.\end{aligned} \tag{52}$$

The coefficients of variables $\{x_e\}$ in (52) are $\lfloor \alpha_e \rfloor + \frac{\hat{\alpha_e}}{\hat{o}}$ if $e \in E^1$ and $\lceil \alpha_e \rceil$ if $e \in E^2$. Similarly, the coefficients of variables $\{y_i\}$ in (52) are $\lfloor \beta_i \rfloor + \frac{\hat{\beta_i}}{\hat{o}}$ if $i \in V^1$ and $\lceil \beta_i \rceil$ if $i \in V^2$ and the coefficients of variables $\{z_{ij}\}$ are $\lfloor \gamma_{ij} \rfloor + \frac{\hat{\gamma_{ij}}}{\hat{o}}$ if $(i,j) \in A^1$ and $\lceil \gamma_{ij} \rceil$ if $(i,j) \in A^2$.

The best choices of coefficients for the sets E^1, V^1 and A^1 are $E^1 = \{e \in E : \hat{\alpha_e} \leq \hat{o}\}$, $V^1 = \{i \in V' : \hat{\beta_i} \leq \hat{o}\}$ and $A^1 = \{(i,j) \in A : \hat{\gamma_{ij}} \leq \hat{o}\}$, respectively.

By defining $\varphi^o(m) = \lfloor m \rfloor + \min\left\{\frac{\hat{m}}{\hat{o}}, 1\right\}$, $m \in \mathbb{R}$, $o \in \mathbb{R}$, $\hat{o} > 0$, inequality (52) becomes inequality (20).□

Theorem 6 *Let us associate penalties $\lambda_i \in \mathbb{R}$, $\forall i \in V_C$, with constraints* (27)*, and $\lambda_i \leq 0$, $\forall i \in V_F$, with constraint* (28)*. For each $i \in V_C$, define $\overline{a}_{i\ell} = a_{i\ell} + \sum_{j \in V_F(R_\ell)} b^j_{i\ell}$, and let $\mathscr{R}_i = \{\ell \in \mathscr{R} : \overline{a}_{i\ell} > 0\}$. For each $i \in V_C$ compute:*

$$\nu_i = q_i \min_{\ell \in \mathscr{R}_i} \left\{ \frac{(c_\ell + p_\ell) - \sum_{j \in V_C} \overline{a}_{j\ell}\lambda_j - \sum_{j \in V_F} a_{j\ell}\lambda_j}{\sum_{j \in V_C} \overline{a}_{j\ell} q_j} \right\} \tag{30}$$

A feasible DSP solution $\mathbf{u}$ *of cost* $z(DSP(\boldsymbol{\lambda}))$ *is given by the following expressions:*

$$u_i = \nu_i + \lambda_i, \forall i \in V_C, \quad \textit{and } u_i = \lambda_i, \forall i \in V_F. \tag{31}$$

Proof. Consider a route $\ell \in \mathscr{R}$. Since $\ell \in \mathscr{R}_i$, $\forall i \in V_C(R_\ell)$, from expression (30) we derive:

$$\nu_i \leq q_i \frac{(c_\ell + p_\ell) - \sum_{j \in V_C} \overline{a}_{j\ell}\lambda_j - \sum_{j \in V_F} a_{j\ell}\lambda_j}{\sum\limits_{j \in V_C} \overline{a}_{j\ell} q_j}, \quad \forall i \in V_C(R_\ell). \tag{53}$$

Given a route $\ell \in \mathscr{R}$, from expression (31) we obtain:

$$\sum_{i \in V_C} \overline{a}_{i\ell} u_i + \sum_{i \in V_F} a_{i\ell} u_i \leq \sum_{i \in V_C} \overline{a}_{i\ell} q_i \frac{(c_\ell + p_\ell) - \sum_{j \in V_C} \overline{a}_{j\ell}\lambda_j - \sum_{j \in V_F} a_{j\ell}\lambda_j}{\sum\limits_{j \in V_C} \overline{a}_{j\ell} q_j} + \sum_{i \in V_C} \overline{a}_{i\ell}\lambda_i + \sum_{i \in V_F} a_{i\ell}\lambda_i. \tag{54}$$

Inequality (54) can be written as:

$$\sum_{i \in V_C} \overline{a}_{i\ell} u_i + \sum_{i \in V_F} a_{i\ell} u_i \leq c_\ell + p_\ell, \tag{55}$$

that corresponds to the constraint of problem DSP for route ℓ.□

Let $E(S)$ denote the set of edges in G with both end-nodes in S and, given two disjoint vertex sets S_1, S_2, let $E(S1 : S2)$ denote the set of edges crossing from S_1 to S_2 (i.e., $E(S_1 : S_2) = \delta(S_1) \cap \delta(S_2)$) (if $S_1 = \{i\}$, we simply write $E(i : S_2)$ instead of $E(\{i\} : S_2)$).

Theorem 7 *The LP-relaxation of the SP formulation satisfies both CI and FrCI inequalities, and a weak form of MI inequalities.*

Proof. Consider a set $S \subseteq V'$ with $V_C(S) \neq \emptyset$ and let $T = V_C(S)$ be the set of customers contained in S. Define the surrogate constraint obtained by adding partitioning constraints (27) corresponding to customers in T after having multiplied the equation associated with $i \in T$ by q_i:

$$\sum_{\ell \in \mathscr{R}} q_\ell(T)\xi_\ell = q(T), \tag{56}$$

where $q(T) = \sum_{i \in S} q_i$ and $q_\ell(T) = \sum_{i \in S} q_i \overline{a}_{i\ell}$. Since $q_\ell(T) \leq \min[Q, q(T)]$, we have

$$\sum_{\ell \in \mathscr{R}(T)} \xi_\ell \geq \max[1, q(T)/Q], \tag{57}$$

where $\mathscr{R}(T) = \{\ell \in \mathscr{R} : \overline{a}_{i\ell} = 1 \text{ for some } i \in T\}$. Given a route $\ell \in \mathscr{R}(T)$, define $\overline{q}_\ell(T)$ as the total demand of the customers not in T assigned to the route, i.e. $\overline{q}_\ell(T) = q_\ell(\overline{T})$, where $\overline{T} = (V_C \setminus T) \cap V_C(R_\ell)$. As $q_\ell(T) + \overline{q}_\ell(T) \leq Q$ we have:

$$\sum_{\ell \in \mathscr{R}(T)} Q\xi_\ell \geq \sum_{\ell \in \mathscr{R}} q_\ell(T)\xi_\ell + \sum_{\ell \in \mathscr{R}} \overline{q}_\ell(T)\xi_\ell. \tag{58}$$

From equations (56) and inequalities (58) we derive:

$$\sum_{\ell \in \mathscr{R}(T)} \xi_\ell \geq \max\left\{1, \frac{1}{Q}\left(q(T) + \sum_{\ell \in \mathscr{R}} \overline{q}_\ell(T)\xi_\ell\right)\right\}. \tag{59}$$

Note that any route $\ell \in \mathscr{R}(T)$ contains at least two edges, one having an ending node in S and the other in $\overline{S}$. Therefore, we have:

$$\sum_{\{i,j\} \in \delta(S)} \eta^\ell_{ij}\xi_\ell \geq 2\xi_\ell. \tag{60}$$

Adding inequality (60) for all $\ell \in \mathscr{R}(T)$ we obtain:

$$\sum_{\ell \in \mathscr{R}(T)} \sum_{\{i,j\} \in \delta(S)} \eta^\ell_{ij}\xi_\ell \geq 2 \sum_{\ell \in \mathscr{R}(T)} \xi_\ell. \tag{61}$$

Thus inequalities (59) become

$$\sum_{\ell \in \mathscr{R}(T)} \rho_\ell(S)\xi_\ell \geq 2\max\left\{1, \frac{1}{Q}\left(q(T) + \sum_{\ell \in \mathscr{R}} \overline{q}_\ell(T)\xi_\ell\right)\right\}, \tag{62}$$

where $\rho_\ell(S) = \sum_{\{i,j\} \in \delta(S)} \eta^\ell_{ij}$. Since

$$\overline{q}_\ell(T) \geq \sum_{j \in \overline{T}} \sum_{k \in V_F(S)} b^k_{j\ell} q_j + \frac{1}{2}\sum_{j \in \overline{T}} q_j \sum_{\{i,h\} \in E(S:\{j\})} \eta^\ell_{ih} \tag{63}$$

from (63) we obtain:

$$\sum_{\ell \in \mathscr{R}(T)} \rho_\ell(S)\xi_\ell \geq 2\max\{1, \frac{1}{Q}(q(T) + \sum_{\ell \in \mathscr{R}}(\sum_{j \in \overline{T}} \sum_{k \in V_F(S)} b^k_{j\ell} q_j + \frac{1}{2}\sum_{j \in \overline{T}} q_j \sum_{\{i,h\} \in E(S:\{j\})} \eta^\ell_{ih})\xi_\ell)\}. \tag{64}$$

We have:

i) $q(T) = \sum_{\ell \in \mathscr{R}} q_\ell(T)\xi_\ell = \sum_{i \in T} q_i(\sum_{\ell \in \mathscr{R}} a_{i\ell}\xi_\ell) + \sum_{i \in T} q_i(\sum_{\ell \in \mathscr{R}} \sum_{k \in V_F(R_\ell)} b^k_{i\ell}\xi_\ell)$;

ii) $\sum_{\ell \in \mathscr{R}} \sum_{j \in \overline{T}} \sum_{k \in V_F(S)} b^k_{j\ell} q_j \xi_\ell = \sum_{j \in \overline{T}} q_j(\sum_{\ell \in \mathscr{R}} \sum_{k \in V_F(S)} b^k_{j\ell}\xi_\ell)$.

Using the equations (32)-(35) linking variables variables ξ with (x, z, w), we derive:

i) $\sum_{i \in T} q_i(\sum_{\ell \in \mathscr{R}} a_{i\ell}\xi_\ell) = \sum_{i \in V_C(S)} q_i y_i$;

ii) $\sum_{i \in T} q_i(\sum_{\ell \in \mathscr{R}} \sum_{k \in V_F(R_\ell)} b^k_{i\ell}\xi_\ell) + \sum_{j \in \overline{T}} q_j(\sum_{\ell \in \mathscr{R}} \sum_{k \in V_F(S)} b^k_{j\ell}\xi_\ell) \geq \sum_{\substack{(i,j) \in A: \\ j \in V_F(S)}} q_i z_{ij}$;

iii) $\frac{1}{2}\sum_{j \in \overline{T}} q_j(\sum_{\ell \in \mathscr{R}}) \sum_{\{i,h\} \in E(S:\{j\})} \eta^\ell_{ih}\xi_\ell = \frac{1}{2}\sum_{i \in V_C(\overline{S})} \sum_{j \in S} q_i x_{\{i,j\}}$.

Using the above equations, from (64) we obtain:

$$\sum_{e \in \delta(S)} x_e \geq 2\max\left\{1, \frac{1}{Q}\left(\sum_{i \in V_C(S)} q_i y_i + \sum_{(i,j) \in A: j \in V_F(S)} q_i z_{ij} + \frac{1}{2}\sum_{i \in V_C(\overline{S})} \sum_{j \in S} q_i x_{\{i,j\}}\right)\right\} \tag{65}$$

□

Theorem 8 *Let (x, z, y) be a solution of the LP-relaxation of formulation TI and assume that $q_i \leq Q$, $\forall i \in V_C$, and that $x_e = 0$, $e = \{i, j\} \in E \setminus \{\{0, h\} : h \in V'\}$, if $q_i + q_j > Q$. The separation problem for MI inequalities* (10) *is solvable in polynomial time.*

Proof. Consider the MI inequality for a given set $S \subseteq V'$, $S \neq \emptyset$:

$$\sum_{e\in\delta(S)} x_e \geq \frac{2}{Q}\left(\sum_{i\in V_C(S)} q_i y_i + \sum_{(i,j)\in A: j\in V_F(S)} q_i z_{ij} + \sum_{i\in V_C(\overline{S})}\sum_{j\in S} q_i x_{\{i,j\}}\right). \tag{66}$$

We have:

$$\sum_{e\in\delta(S)} x_e = \sum_{e\in E(0:S)} x_e + \sum_{e\in E(S:\overline{S})} x_e, \tag{67}$$

and for each $i \in V'$ (see equation (2)):

$$\sum_{e\in\delta(i)} x_e = 2y_i = x_{\{0,i\}} + \sum_{e\in E(i:S)} x_e + \sum_{e\in E(i:\overline{S})} x_e. \tag{68}$$

From equation (68), the term $\sum_{i\in V_C(S)} q_i y_i$ of inequality (66) can be rewritten as follows:

$$\sum_{i\in V_C(S)} q_i y_i = \sum_{i\in V_C(S)} \frac{q_i}{2}\left(x_{\{0,i\}} + \sum_{e\in E(i:S)} x_e + \sum_{e\in E(i:\overline{S})} x_e\right). \tag{69}$$

The MI inequality (66) can be rewritten as:

$$\begin{aligned}\sum_{e\in E(0:S)} x_e + \sum_{e\in E(S:\overline{S})} x_e \geq \sum_{i\in V_C(S)} \frac{q_i}{Q} x_{\{0,i\}} + \sum_{i\in V_C(S)}\sum_{e\in E(i:S)} \frac{q_i}{Q} x_e + \\ \sum_{i\in V_C(S)}\sum_{e\in E(i:\overline{S})} \frac{q_i}{Q} x_e + \frac{2}{Q}\sum_{\substack{(i,j)\in A:\\ j\in V_F(S)}} q_i z_{ij} + \sum_{i\in V_C(\overline{S})}\sum_{j\in S}\frac{q_i}{Q} x_{\{i,j\}} + \sum_{i\in V_C(\overline{S})}\sum_{j\in S}\frac{q_i}{Q} x_{\{i,j\}}.\end{aligned} \tag{70}$$

We also have:

$$\sum_{e\in E(0:S)} x_e = \sum_{e\in E(0:V_C(S))} x_e + \sum_{e\in E(0:S\setminus V_C(S))} x_e, \tag{71}$$

and

$$\sum_{e\in E(S:\overline{S})} x_e = \sum_{j\in S}\sum_{\{i,j\}\in E(j:V_C(\overline{S}))} x_{\{i,j\}} + \sum_{j\in S}\sum_{\{i,j\}\in E(j:(V'\setminus V_C(\overline{S}))\setminus S)} x_{\{i,j\}}. \tag{72}$$

Notice that $S \setminus V_C(S) = V_F(S)$ and that $(V' \setminus V_C(\overline{S})) \setminus S = V_F(\overline{S})$. Then, inequality (70) can be rewritten as:

$$\begin{aligned}\sum_{e\in E(0:V_C(S))} x_e + \sum_{e\in E(0:V_F(S))} x_e + \sum_{j\in S}\sum_{\{i,j\}\in E(j:V_C(\overline{S}))} x_{\{i,j\}} + \sum_{j\in S}\sum_{\{i,j\}\in E(j:V_F(\overline{S}))} x_{\{i,j\}} \geq \\ \sum_{i\in V_C(S)} \frac{q_i}{Q} x_{\{0,i\}} + \sum_{i\in V_C(S)}\sum_{e\in E(i:S)} \frac{q_i}{Q} x_e + \sum_{i\in V_C(S)}\sum_{e\in E(i:\overline{S})} \frac{q_i}{Q} x_e + \\ \frac{2}{Q}\sum_{\substack{(i,j)\in A:\\ j\in V_F(S)}} q_i z_{ij} + \sum_{i\in V_C(\overline{S})}\sum_{j\in S}\frac{q_i}{Q} x_{\{i,j\}} + \sum_{i\in V_C(\overline{S})}\sum_{j\in S}\frac{q_i}{Q} x_{\{i,j\}}.\end{aligned} \tag{73}$$

Notice that as $q_i = 0$, $\forall i \in V_F$, we have:

$$\sum_{i\in V_C(S)}\sum_{e\in E(i:\overline{S})} \frac{q_i}{Q} x_e = \sum_{i\in S}\sum_{e\in E(i:\overline{S})} \frac{q_i}{Q} x_e = \sum_{i\in S}\sum_{e\in E(i:V_C(\overline{S}))} \frac{q_i}{Q} x_e + \sum_{i\in S}\sum_{e\in E(i:V_F(\overline{S}))} \frac{q_i}{Q} x_e, \tag{74}$$

$$\sum_{i \in V_C(S)} \sum_{e \in E(i:S)} \frac{q_i}{Q} x_e = \sum_{i \in S} \sum_{e \in E(i:S)} \frac{q_i}{Q} x_e, \tag{75}$$

and

$$\sum_{i \in V_C(\overline{S})} \sum_{j \in S} \frac{q_i}{Q} x_{\{i,j\}} = \sum_{i \in \overline{S}} \sum_{j \in S} \frac{q_i}{Q} x_{\{i,j\}}. \tag{76}$$

Inequality (73) can be rewritten as:

$$\begin{aligned} &\sum_{\{0,i\} \in E(0:V_C(S))} (1 - q_i/Q) x_{\{0,i\}} + \sum_{e \in E(0:V_F(S))} x_e + \sum_{j \in S} \sum_{\{i,j\} \in E(j:V_C(\overline{S}))} (1 - (q_i + q_j)/Q) x_{\{i,j\}} + \\ &\sum_{j \in S} \sum_{\{i,j\} \in E(j:V_F(\overline{S}))} (1 - q_j/Q) x_{\{i,j\}} \geq \sum_{i \in S} \sum_{e \in E(i:S)} \frac{q_i}{Q} x_e + \frac{2}{Q} \sum_{\substack{(i,j) \in A: \\ j \in V_F(S)}} q_i z_{ij} + \sum_{i \in \overline{S}} \sum_{j \in S} \frac{q_i}{Q} x_{\{i,j\}}. \end{aligned} \tag{77}$$

Since

$$\begin{aligned} &\sum_{i \in S} \sum_{e \in E(i:S)} \frac{q_i}{Q} x_e + \sum_{i \in \overline{S}} \sum_{j \in S} \frac{q_i}{Q} x_{\{i,j\}} = \sum_{j \in S} \left(\sum_{\substack{\{j,i\} \in \delta(j): \\ j < i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\} \in \delta(j): \\ i < j}} \frac{q_i}{Q} x_{\{i,j\}} \right) = \\ &\sum_{j \in V'} \left(\sum_{\substack{\{j,i\} \in \delta(j): \\ j < i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\} \in \delta(j): \\ i < j}} \frac{q_i}{Q} x_{\{i,j\}} \right) - \sum_{j \in \overline{S}} \left(\sum_{\substack{\{j,i\} \in \delta(j): \\ j < i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\} \in \delta(j): \\ i < j}} \frac{q_i}{Q} x_{\{i,j\}} \right), \end{aligned} \tag{78}$$

and

$$\begin{aligned} \sum_{\substack{(i,j) \in A: \\ j \in V_F(S)}} q_i z_{ij} = q(V_C) - \sum_{i \in V_C(S)} q_i y_i - \sum_{i \in V_C(\overline{S})} q_i y_i - \sum_{\substack{(i,j) \in A: \\ j \in V_F(\overline{S})}} q_i z_{ij} = \\ q(V_C) - \sum_{i \in V_C} q_i y_i - \sum_{\substack{(i,j) \in A: \\ j \in V_F(\overline{S})}} q_i z_{ij}, \end{aligned} \tag{79}$$

inequality (77) can be rewritten as:

$$\begin{aligned} &\sum_{\{0,i\} \in E(0:V_C(S))} (1 - q_i/Q) x_{\{0,i\}} + \sum_{e \in E(0:V_F(S))} x_e + \sum_{j \in S} \sum_{\{i,j\} \in E(j:V_C(\overline{S}))} (1 - (q_i + q_j)/Q) x_{\{i,j\}} + \\ &\sum_{j \in \overline{S}} \left(\sum_{\substack{\{j,i\} \in \delta(j): \\ j < i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\} \in \delta(j): \\ i < j}} \frac{q_i}{Q} x_{\{i,j\}} \right) + \sum_{j \in S} \sum_{\{i,j\} \in E(\{j\}:V_F(\overline{S}))} (1 - q_j/Q) x_{\{i,j\}} + \\ &\frac{2}{Q} \sum_{\substack{(i,j) \in A: \\ j \in V_F(\overline{S})}} q_i z_{ij} \geq \sum_{j \in V'} \left(\sum_{\substack{\{j,i\} \in \delta(j): \\ j < i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\} \in \delta(j): \\ i < j}} \frac{q_i}{Q} x_{\{i,j\}} \right) + \frac{2}{Q} (q(V_C) - \sum_{i \in V_C} q_i y_i). \end{aligned} \tag{80}$$

Notice that, as $q_i \leq Q$, $\forall i \in V_C$, $x_e = 0$, $e = \{i,j\} \in E \setminus \{\{0,h\} : h \in V'\}$, if $q_i + q_j > Q$, all the variable coefficients of the above inequality are nonnegative whereas the right-hand-side of the inequality does not depend on the set S.

The most violated constraint (80) can now be found by computing a minimum s-t cut on an directed capacitated graph $\overline{G} = (\overline{V}, \overline{A})$ with $\overline{V} = V' \cup \{s,t\}$ and $\overline{A} = \{(i,j),(j,i) : \forall \{i,j\} \in E \setminus \{\{0,j\} : j \in V'\}\} \cup \{(s,i) : \{0,i\} \in E\} \cup \{(i,t) : i \in V'\}$. The additional nodes s and t represent source and sink node, respectively. The arcs capacities are defined as follows:

- Every arc (s, i), $i \in V_C$ is associated with a capacity $(1 - q_i/Q)x_{\{0,i\}}$;
- Every arc (s, i), $i \in V_F$ is associated with a capacity $x_{\{0,i\}}$;
- Every arc (i, j), $i \in V_C$, $j \in V'$ is associated with a capacity $(1 - (q_i + q_j)/Q)x_{\{i,j\}}$;
- Every arc (i, j), $i \in V_F$, $j \in V'$ is associated with a capacity $(1 - q_j)x_{\{i,j\}}$;
- Every arc (j, t), $j \in V_C$, is associated with a capacity $\sum_{\substack{\{j,i\}\in\delta(j):\\ j<i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\}\in\delta(j):\\ i<j}} \frac{q_i}{Q} x_{\{i,j\}}$;
- Every arc (j, t), $j \in V_F$, is associated with a capacity $\frac{2}{Q} \sum_{\substack{(i,j)\in A:\\ j\in V_F(\overline{S})}} q_i z_{ij} + (\sum_{\substack{\{j,i\}\in\delta(j):\\ j<i}} \frac{q_i}{Q} x_{\{j,i\}} + \sum_{\substack{\{i,j\}\in\delta(j):\\ i<j}} \frac{q_i}{Q} x_{\{i,j\}})$.

Let $(\overline{S}, \overline{V} \setminus \overline{S})$ be the minimum s-t cut of $\overline{G}$ and assume that $t \in \overline{S}$. One can see that if the cut capacity is strictly smaller than right-hand-side of inequality (80) then node set $S = \overline{S} \setminus \{t\}$ defines the most violated inequality (80). No violated inequality exists if the cut has a capacity greater than or equal to the value of the right-hand-side of (80). □

10 Details of the heuristic algorithms

10.1 A constructive heuristic

Given an instance of VRPTF, we define a complete graph $\overline{G} = (\overline{V}, \overline{E})$ where the node set $\overline{V} = \{0\} \cup V_C$ contains the depot and the customer nodes. Each edge $e \in \overline{E}$ has a cost given by r_e. Each customer $i \in V_C$ has a demand equal to q_i and the capacity of the vehicles is set to Q. Let $m = \lceil \sum_{i \in V_c} q_i/Q \rceil$ be a lower bound on the minimum number of routes required. The details of our implementation of the three phases are as following.

(i) m "seed" customers are randomly selected to initialize the m routes of the emerging CVRP solution. The remaining customers are partitioned into m subsets by heuristically solving a Generalized Assignment Problem (GAP) where each bin k is associated with the k-th customer selected to initialize a route. The assignment cost a_{ik} for allocating customer i to bin k is $r_{0i} + \alpha r_{\{i,k\}} - \beta r_{0k}$, where α, β are nonnegative parameters. The GAP is solved heuristically. If, for a given m, the GAP solution is infeasible, we set $m = m + 1$ and we repeat the above procedure.

(ii) The route for a subset of customers is determined by solving a TSP on the subgraph induced by the subset. We apply a 3-opt procedure to a starting tour obtained by generating a random sequence of the customers.

(iii) The solution obtained at step (ii) is locally optimized using a classical multiroute improvement procedure consisting of two types of operations: (a) movement of a customer from one route to another; (b) exchange of two customers belonging to different routes. We try all possible such operations until no improvement can be obtained. Each route of the modified solution is then re-optimized with the 3-opt procedure.

Since the initial partitioning and the TSP solutions are based on random choices, we can obtain different solutions executing the three phases several times. In our implementation we run them 2000 times: for the first 1000 runs we set $\alpha = 1.1$ and $\beta = 0.7$ (see step (i)), while in the last 1000 runs we set $\beta = 0$, leaving α unchanged.

The CVRP solution so far obtained, is optimized by iteratively applying two re-optimization procedures: procedure Squeeze used by Baldacci et al. (2007) for the CmRSP, and procedure LS-multiple. The two procedures are repeated in sequence until the current solution can be improved. Procedure Squeeze tries to re-optimize the routing (i.e. the set $\overline{E}$) by allowing a few changes in the customer connections (set $\overline{A}$). Procedure LS-multiple is a multiroute improvement procedure based on customer exchanges among the routes of the current solution.

10.2 A Lagrangean heuristic

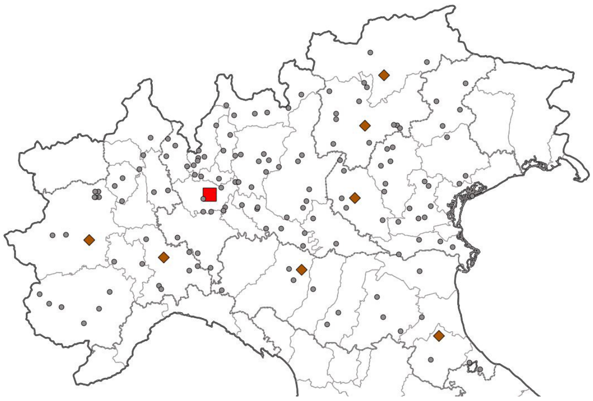
Procedure CG is interwoven with a heuristic algorithm that produces a feasible VRPTF solution of cost $\hat{z}$ using the route sets $\tilde{\mathscr{R}}$ (see Step 2 of procedure CG). Given the current $DSP(\boldsymbol{\lambda})$ solution, define vector $\tilde{\boldsymbol{\xi}}$ as follows:

$$\tilde{\xi}_\ell = \sum_{i \in V_C} \overline{a}_{i\ell} \frac{q_i}{q(R_\ell)} \zeta^i_\ell, \quad \ell \in \tilde{\mathscr{R}}, \tag{81}$$

by setting $\zeta^i_{\ell(i)} = 1$ and $\zeta^i_\ell = 0$, $\forall \ell \in \tilde{\mathscr{R}} \setminus \{\ell(i)\}$, $\forall i \in V_C$. Define $C(\ell) = V_C(R_\ell) \cup V_A(R_\ell)$, i.e. $C(\ell)$ is the set of customers either visited on the route or assigned to facilities in $V_F(R_\ell)$. The heuristic algorithm performs the following steps.

1. *Initialization.* Initialize $\hat{z} = 0$, $SOL = \emptyset$ and $\delta(i) = 0$, and $\forall i \in V'$.
2. *Extract a subset of routes $SOL \subseteq \tilde{\mathscr{R}}$.* Let ℓ^* be the route of $\tilde{\mathscr{R}}$ where $\tilde{\xi}_{\ell^*} = \max\{\tilde{\xi}_\ell : \ell \in \tilde{\mathscr{R}}\}$. Remove ℓ^* from $\tilde{\mathscr{R}}$. If $\delta(i) = 0$, for some $i \in C(\ell^*)$, then update $SOL = SOL \cup \{\ell^*\}$, $\delta(i) = \delta(i) + \overline{a}_{i\ell}$, $\forall i \in C(\ell^*)$, and $\delta(i) = \delta(i) + a_{i\ell}$, $\forall i \in V_F(R^*_\ell)$. Repeat step 2 until $\tilde{\mathscr{R}} = \emptyset$.
3. *Modify the route set SOL so that $\delta(i) \leq 1$, $\forall i \in V'$.*
   a) Remove from SOL any route $\ell \in SOL$ such that $\delta(i) > 1$, $\forall i \in C(\ell^*)$, and update $\delta(i)$, $\forall i \in V'$, accordingly. For each $\ell \in SOL$, compute the savings that can be achieved by removing from route ℓ every customer $i \in C(\ell^*)$ having $\delta(i) > 1$. Let $\ell^* \in SOL$ be the route of maximum saving. Remove from route ℓ^* every customer $i \in C(\ell^*)$ with $\delta(i) > 1$, and update $\delta(i)$. Repeat step 3.a until $\delta(i) \leq 1$, for each $i \in V_C$.
   b) For each $\ell \in SOL$, compute the total number $\alpha(\ell)$ of customers assigned to every facility $i \in V_F(R^*_\ell)$ having $\delta(i) > 1$. Let ℓ^* be the route having the minimum $\alpha(\ell)$ value. Remove from route ℓ^* every facility $i \in V_F(R_{\ell^*})$ with $\delta(i) > 1$, update $\delta(i)$, and $\delta(j)$, $\forall j \in V_C$, accordingly. Repeat step 3.b until $\delta(i) \leq 1$, for each $i \in V_F$.
   c) For each $\ell \in SOL$, remove any facility $i \in V_F(R_\ell)$ with $\delta(i) = 1$ and without customers assigned to it, and update $\delta(i) = \delta(i) - 1$.
4. *Insert unrouted customers.* For each unrouted customer i (i.e., $\delta(i) = 0$) perform the following operations. Compute the minimum extra-cost $exc(i, \ell)$ for inserting i in route $\ell \in SOL$ without considering assignment of i to facilities in $V_F(R_\ell)$. We set $exc(i, \ell) = \infty$ if the total load of the resulting route ℓ exceeds the vehicle capacity Q. Let ℓ^* be such that $exc(i, \ell^*) = \min_{\ell \in SOL}[exc(i, \ell)]$. If $exc(i, \ell^*) = \infty$, then set $\hat{z} = \infty$ and stop; otherwise, insert customer i in route ℓ^* in the position of cost $exc(i, \ell^*)$ and set $\delta(i) = 1$.
5. *Define the* VRPTF *solution $\boldsymbol{\xi}$.* Define $\xi_\ell = 1$, for each $\ell \in SOL$, and $\xi_\ell = 0$, for each $\ell \in \mathscr{R} \setminus SOL$.
6. *Local optimization.* Locally optimize solution $\boldsymbol{\xi}$ by iteratively applying the two re-optimization procedures Squeeze and LS-multiple used also for the constructive heuristic.

Figure 1: Real-world instances: North area



11 Details about the instances

11.1 Real-world based instances

A set of six test instances for each area (*North*, *Centre* and *South*) were generated by the company based on the following settings.

- The set of customers is selected from the customers that are currently served on a daily basis by using different criteria. The number of customers varies from a minimum of 54 up top a maximum of 164. The customer demands were computed based on historical data;
- The set of facilities corresponds to the existing set of facilities and can also include new facilities that the company want evaluate in order to revise the current distribution network. Instances with 4, 6, 7, 9, 12, 13, and 18 facilities were generated;
- Two types of fleet of vehicles were considered: vehicles with capacity equal to 24 pallets (*single-unit 3 axes* type of trucks) and vehicles with capacity equal to 36 pallets (*single-trailer 3 axes* type of trucks), respectively;
- The routing cost of a pair of nodes i and j of the network were computed as $r_{\{i,j\}} = c\ dist_{ij}$ where $dist_{ij}$ represents the distance in kilometer between nodes i and j computed using a digital map of the Italian territory, and c is the routing cost per kilometers (currency in expressed in Euro (€)) associated with the type of vehicle (either *single-unit* or *single-trailer*);
- Set $\{F_i\}$ of facilities to which the customers can be assigned are defined directly by the company using different criteria. These criteria take into account the customer demand, required level of service, a priori agreements between the customers and the company, and the distance matrix $[d_{ij}]$ used to compute the routing costs;
- The distribution from the facilities to the customers is performed by means of a fleet of *single-unit 2 axes* type of trucks with a vehicle capacity ranging from 6 to 8 pallets. The distribution cost from the facilities depends on the type of contract that has been defined between the company and the third-party contractor and vary from facility to facility. The distribution cost is a function of the number of pallets associated with the order and the distance between the customer location and the facility. Therefore the assignment cost matrix is defined by the company using the current distribution tariff agreed with the third-party companies.

Figure 2: Real-world instances: Centre area

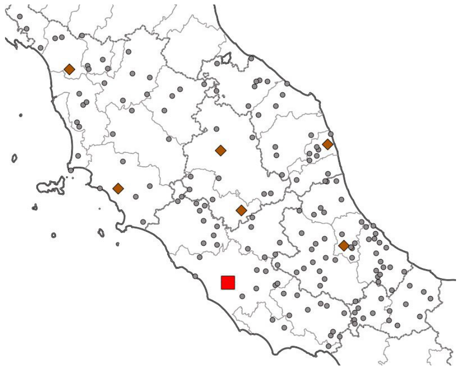


Figure 3: Real-world instances: South area

A total number of 18 instances were generated, 6 instances per areas or depots. Figures 11.1, 11.1, and 11.1 illustrate the layout of the three distributions ares. In the figure, the three depots are represented with squares, and rhombus and circles represent facilities and customers, respectively.

11.2 LRP based instances

From each LRP instance we derived a VRPTF instance as follows.

i) The set V_F of facilities, the set V_C of customers (and the associated demands), correspond to the set of depots and customers of the original LRP instance;

ii) The vehicle capacity Q is equal to the vehicle capacity of the original LRP instance;

iii) The depot coordinates were defined as follows. Let x_{min} and x_{max} be the minimum and maximum x-coordinates among the customers and the facilities x-coordinate, respectively; similarly define

y_{min} and y_{max}. The coordinate (x, y) of the central depot are defined as follows:

$$x = x_{min} + \lfloor (x_{max} - x_{min})/2 \rfloor \quad \text{and} \quad y = y_{min} + \lfloor (y_{max} - y_{min})/2 \rfloor. \tag{82}$$

The routing and connection costs were generated as follows.

Class A. Routing and assignment costs of a pair of nodes i, j are equal to the Euclidean distance e_{ij}, computed according to the TSPLIB EUC_2D standard.

Class B. For each pair of nodes i, j, the routing cost is $r_{\{i,j\}} = \lfloor \alpha e_{ij} \rfloor$, while the assignment cost is $d_{ij} = \lfloor (10 - \alpha) e_{ij} \rfloor$, where $\alpha = 7.0$.

For all the instances, every customer can be assigned to every facility, i.e. $F_i = V_F$, $\forall i \in V_C$.

We generated a total number of 150 instances, 75 instances per class. The dimensions of the instances vary from very small instances with 12 customers and two facilities up to very large instances with 150 customers and 20 facilities.

12 Details about the computational results on LRP based instances

This section reports the complete details about the computational results on LRP based instances.

Table 6: Results on Class A: Akca et al. (2009) LRP based instances

Name	z^*	#r	#f	#c	%UB_1	%UB_2	%LB_1	%LB_2	t_{DA}	%LB_C	t_C	%LB	t_{LB}	#cuts	#cols	#N	%Opt	t_{TOT}
cr30x5a-1	621	5	0	0	100.0	100.0	99.4	100.0	2.6	98.8	2.7	100.0	0.1	0	303	1	100.0	8
cr30x5a-2	665	5	1	2	100.2	102.1	90.6	94.0	2.8	97.5	2.5	99.1	3.2	275	2185	23	100.0	20
cr30x5a-3	575	5	1	2	100.7	100.0	90.8	96.5	2.5	98.5	2.2	98.5	1.3	252	1160	93	100.0	29
cr30x5b-1	727	5	1	1	100.0	100.0	96.4	98.5	2.9	99.7	2.9	100.0	1.3	479	959	1	100.0	10
cr30x5b-2	826	6	0	0	100.0	100.6	92.5	93.6	1.5	93.0	3.7	97.6	2.1	78	1609	367	100.0	79
cr30x5b-3	788	7	0	0	100.1	100.0	94.6	96.1	32.5	94.2	2.8	97.6	1.9	94	1096	1335	100.0	1061
cr40x5a-1	738	7	3	8	100.0	100.0	95.5	97.5	19.5	94.9	4.2	97.9	2.5	98	1530	151	100.0	82
cr40x5a-2	786	6	1	1	100.5	101.0	92.7	96.9	5.2	95.8	8.3	98.2	3.8	357	1627	986	99.3	3615
cr40x5a-3	807	6	4	4	101.1	101.5	94.8	98.0	21.4	95.2	5.3	98.5	2.8	405	1212	562	98.8	3631
cr40x5b-1	964	8	0	0	100.0	101.0	95.9	98.1	11.1	95.6	6.0	98.9	1.3	238	646	187	100.0	49
cr40x5b-2	901	8	2	3	100.0	100.1	93.1	95.7	3.3	95.5	7.9	98.1	3.4	372	910	41	100.0	34
cr40x5b-3	887	8	2	5	100.5	100.5	95.2	96.7	13.5	95.0	6.4	98.1	1.5	295	876	378	100.0	81

Table 7: Results on Class A: Prins et al. (2004) LRP based instances

Name	z^*	#r	#f	#c	%UB_1	%UB_2	%LB_1	%LB_2	t_{DA}	%LB_C	t_C	%LB	t_{LB}	#cuts	#cols	#N	%Opt	t_{TOT}
ppw-20-5-0-a	253	5	1	1	100.0	100.0	95.2	96.7	1.5	98.5	1.2	100.0	0.2	11	394	1	100.0	5
ppw-20-5-0-b	211	3	1	2	100.0	100.0	85.3	94.4	7.8	100.0	0.5	97.8	0.6	120	803	17	100.0	12
ppw-20-5-2-a	247	5	1	3	100.0	100.0	90.8	94.5	3.2	95.4	1.3	97.4	0.4	157	711	197	100.0	14
ppw-20-5-2-b	189	3	1	2	100.0	100.0	85.7	93.1	1.6	100.0	0.3	100.0	1.1	181	2212	1	100.0	5
ppw-50-5-0-a	616	12	1	1	100.2	100.2	97.0	98.9	1.6	93.9	14.8	98.9	2.3	224	794	332	100.0	87
ppw-50-5-0-b	400	6	1	2	100.0	102.0	91.8	95.6	6.3	96.8	9.8	96.7	11.9	362	2964	15	96.9	3621
ppw-50-5-2'-a	653	12	0	0	100.5	100.0	95.9	96.1	1.6	95.6	13.4	99.5	2.6	323	827	85	100.0	48
ppw-50-5-2'-b	351	6	0	0	100.3	100.0	91.1	92.9	5.3	99.0	10.6	99.4	6.9	973	2391	204	99.7	3619
ppw-50-5-2-a	587	12	1	2	100.0	100.0	96.7	98.3	1.5	93.6	11.1	98.9	2.1	828	880	949	100.0	126
ppw-50-5-2-b	357	6	0	0	100.0	100.3	92.3	93.4	11.3	96.7	13.3	96.7	6.5	514	2174	61	97.1	3625
ppw-50-5-3-a	586	12	1	3	100.2	100.0	94.8	95.4	20.5	92.2	12.4	95.9	2.0	152	728	20734	97.9	3630
ppw-50-5-3-b	381	6	0	0	100.0	100.0	91.2	94.4	28.3	95.9	8.4	96.8	6.4	493	3335	395	97.4	3644
ppw-100-5-0-a	1158	25	2	2	100.9	100.9	97.7	98.8	3.1	93.8	216.9	99.3	10.6	12	1417	1312	100.0	507
ppw-100-5-0-b	679	11	3	3	100.0	100.0	91.2	95.3	146.6	91.6	115.2	96.7	42.6	2019	5451	7	96.8	3819
ppw-100-5-2-a	1010	24	1	1	100.6	100.4	96.8	97.4	3.1	91.7	193.0	97.5	14.0	640	1672	5639	97.8	3735
ppw-100-5-2-b	569	12	1	1	100.0	100.2	94.8	95.6	99.5	90.0	110.1	96.0	30.8	1040	3019	302	96.2	3783
ppw-100-5-3-a	1068	24	1	1	100.7	100.7	97.1	98.2	29.1	93.6	146.8	98.6	8.8	561	1312	8235	98.9	3686
ppw-100-5-3-b	612	11	0	0	100.0	100.7	92.7	95.6	55.7	93.8	131.5	97.1	27.9	1105	5207	183	97.4	3733
ppw-100-10-0-a	1215	24	1	1	100.1	100.7	97.9	98.7	47.9	89.9	154.4	98.9	11.2	372	1040	2420	100.0	991
ppw-100-10-0-b	693	11	1	1	100.0	101.9	94.2	96.6	208.8	93.2	159.1	97.4	38.3	1006	4232	84	97.6	3869
ppw-100-10-2-a	1030	24	0	0	100.8	100.0	96.2	97.7	134.6	90.6	163.8	97.7	9.9	85	1182	6064	98.2	3853
ppw-100-10-2-b	582	11	0	0	100.0	100.9	93.4	95.7	127.1	91.2	133.3	96.4	54.6	829	4798	30	96.5	3809
ppw-100-10-3-a	1055	24	4	6	100.0	100.3	97.6	98.3	121.6	88.7	142.1	99.3	14.8	30	1327	245	100.0	424
ppw-100-10-3-b	608	11	1	2	100.0	101.6	91.2	92.9	98.7	89.7	123.0	94.6	46.8	919	6930	24	94.8	3773

Table 8: Results on Class A: different authors LRP based instances

Name	z^*	$\#r$	$\#f$	$\#c$	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_C$	t_C	$\%LB$	t_{LB}	$\#cuts$	$\#cols$	#N	$\%Opt$	t_{TOT}
Christ-50x5	514	5	2	2	100.6	103.1	95.7	97.3	58.1	98.2	8.3	99.2	7.9	1086	3231	54	100.0	130
Christ-50x5_B	533	5	1	2	100.2	100.4	92.5	95.9	84.8	96.4	7.9	96.8	7.7	592	4284	175	97.2	3698
Christ-75x10	783	9	3	6	100.0	100.0	93.4	94.6	40.0	93.6	58.2	97.1	28.6	665	6206	22	97.3	3676
Christ-75x10_B	814	9	3	5	100.0	101.8	94.0	95.2	112.9	94.2	46.0	97.5	33.3	1096	6417	235	97.8	3750
Christ-100x10	831	8	0	0	100.0	101.4	92.9	93.8	321.8	94.8	63.6	96.8	126.1	1335	17779	5	96.9	3987
Gaskell-21x5	371	4	1	2	100.0	100.0	97.1	98.7	1.3	98.1	1.3	100.0	0.3	157	534	1	100.0	5
Gaskell-22x5	554	3	3	4	102.0	100.0	82.3	88.6	83.7	97.3	1.0	99.8	46.0	301	3410	5	100.0	145
Gaskell-29x5	503	4	1	1	102.2	100.0	88.3	94.0	119.1	93.7	1.6	98.0	67.1	223	1936	67	100.0	683
Gaskell-32x5-2	427	3	0	0	100.0	100.0	92.4	98.9	459.5	100.0	1.4	100.0	97.3	6	5890	1	100.0	567
Gaskell-32x5	479	4	1	1	100.0	100.0	91.7	95.7	224.3	98.7	2.4	100.0	49.8	330	2149	1	100.0	280
Gaskell-36x5	411	4	1	1	100.2	100.2	96.4	96.7	8.8	99.0	3.7	100.0	1.3	140	1875	1	100.0	17
Min-27x5	3083	4	1	1	100.0	100.0	89.4	95.2	17.1	99.2	1.7	100.0	1.5	364	1357	1	100.0	24
Perl83-12x2	100	2	0	0	100.0	100.0	92.1	99.3	0.8	100.0	0.1	100.0	0.1	25	417	1	100.0	2
Perl83-55x15	453	10	3	3	101.5	101.3	96.8	97.9	38.2	94.6	17.5	99.3	3.5	324	2422	467	100.0	278
Perl83-85x7	618	11	1	1	100.2	101.1	96.9	97.7	36.1	92.7	59.5	98.2	13.0	580	3516	1835	98.8	3736
P111112-100x10	1346	11	0	0	100.0	100.6	92.1	94.5	163.2	92.2	112.2	95.4	46.4	976	7651	196	95.6	3846
P111122-100x20	1252	11	1	2	100.0	102.6	93.8	96.1	453.1	93.3	148.3	98.7	58.2	70	5927	625	98.8	4138
P111212-100x10	1266	10	0	0	100.0	100.6	92.8	95.8	46.8	93.1	118.8	96.8	57.4	1144	4786	5	96.9	3718
P111222-100x20	1338	11	1	1	100.0	100.4	91.5	94.0	379.0	90.8	208.6	96.1	74.5	1542	5446	19	96.1	4053
P112112-100x10	1236	11	3	3	100.0	100.0	89.6	93.2	196.6	93.1	173.0	96.7	134.8	2344	9884	15	96.8	3889
P112122-100x20	1047	10	3	3	100.0	100.0	84.6	86.8	485.6	92.4	278.6	94.2	227.5	2348	16311	3	94.2	4177
P112212-100x10	892	11	2	2	100.4	100.0	89.0	90.6	220.3	88.6	107.2	92.1	83.5	1045	9920	15	92.1	3918
P112222-100x20	1006	10	1	1	100.0	103.0	93.4	94.4	93.4	94.7	170.8	95.7	176.1	1039	7578	8	95.7	3747
P113112-100x10	1158	11	0	0	100.0	102.9	89.5	91.8	319.1	93.8	176.5	94.0	58.5	1295	6782	8	94.2	4007
P113122-100x20	1190	11	4	6	100.0	102.3	87.8	90.3	227.5	92.9	221.0	96.1	163.0	2200	10465	13	96.1	3914
P113212-100x10	1154	10	1	1	100.0	104.9	92.9	93.6	48.9	93.1	125.8	95.2	63.8	927	4778	97	95.3	3717
P113222-100x20	1078	11	0	0	100.0	100.0	90.6	91.8	73.2	94.4	235.0	94.3	101.7	1164	5276	68	94.5	3748
P131112-150x10	1833	16	1	1	100.8	100.0	93.7	95.1	171.8	90.2	669.4	95.6	134.3	1112	7283	37	95.7	3946
P131122-150x20	1769	16	1	1	100.0	100.8	92.5	95.5	579.3	89.8	872.2	95.9	137.6	34	9383	56	95.9	4411
P131212-150x10	1802	16	1	2	100.0	101.2	93.5	96.2	318.2	91.1	500.7	97.1	171.2	2076	9637	18	97.2	4147
P131222-150x20	1802	15	2	2	100.0	100.5	93.2	95.2	600.7	89.8	892.7	95.6	119.4	870	7002	87	95.7	4371
P132112-150x10	1783	16	3	3	100.0	101.0	91.8	95.0	969.1	93.3	1147.1	96.5	286.1	3288	11779	60	96.7	4815
P132122-150x20	1541	15	1	1	100.0	100.1	88.8	90.4	731.2	91.0	1240.3	93.8	429.3	3329	23858	2	93.8	4508
P132212-150x10	1251	16	0	0	100.0	101.0	91.5	92.6	240.4	90.8	766.6	94.3	180.1	2126	11721	13	94.3	4073
P132222-150x20	1184	16	1	1	100.0	100.0	92.7	93.7	513.2	89.9	985.7	95.8	514.0	2487	11655	3	95.8	4368
P133112-150x10	1899	16	2	2	100.0	103.2	92.4	93.8	617.6	93.0	1182.5	94.7	409.5	2545	12407	6	94.8	4434
P133122-150x20	1498	16	2	2	100.0	100.8	92.5	93.4	1061.3	90.8	755.5	94.5	190.7	1966	9534	88	94.6	4908
P133212-150x10	1245	16	1	1	100.0	103.9	93.1	94.1	649.9	93.4	818.1	95.6	108.3	1602	7306	39	95.7	4510
P133222-150x20	1551	16	0	0	100.0	100.1	90.3	91.0	836.0	89.3	1069.9	91.5	321.5	1317	10764	222	91.6	4688

Table 9: Results on Class B: Akca et al. (2009) LRP based instances

Name	z^*	$\#r$	$\#f$	$\#c$	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_C$	t_C	$\%LB$	t_{LB}	$\#cuts$	$\#cols$	#N	$\%Opt$	t_{TOT}
cr30x5a-1	4176	5	2	8	100.0	100.2	98.8	99.2	3.4	97.0	2.6	99.5	1.9	426	770	13	100.0	13
cr30x5a-2	4428	5	1	6	100.6	103.3	89.9	92.8	3.2	96.3	3.2	98.2	5.2	338	1431	50	98.3	3609
cr30x5a-3	3655	5	2	11	101.8	100.5	91.5	96.7	3.2	98.7	2.4	98.2	3.3	633	1610	34	100.0	24
cr30x5b-1	4844	5	1	5	102.4	100.0	94.7	97.1	3.0	97.3	3.0	98.4	5.9	893	1141	22	100.0	26
cr30x5b-2	4931	6	4	13	103.3	102.0	95.9	97.2	4.1	95.2	3.1	98.6	0.9	236	656	100	100.0	28
cr30x5b-3	4626	6	3	15	108.3	100.6	95.4	96.5	1.7	94.5	2.1	96.6	0.4	1	707	3330	98.9	3605
cr40x5a-1	4221	6	4	21	100.0	100.5	95.4	96.8	3.5	94.6	5.1	97.4	3.0	15	2112	90	100.0	50
cr40x5a-2	4804	6	4	17	102.0	101.4	91.6	96.9	4.3	94.6	7.1	97.3	5.4	2647	919	282	98.3	3616
cr40x5a-3	4577	6	4	25	106.9	103.9	94.9	97.4	7.4	96.7	3.7	98.8	10.4	1546	1605	213	100.0	2085
cr40x5b-1	6334	9	3	13	102.5	100.5	94.1	96.3	15.2	92.0	5.4	97.0	2.4	592	993	175	100.0	74
cr40x5b-2	5933	8	3	14	100.6	102.1	95.3	96.8	7.5	93.4	6.3	97.2	2.4	219	930	362	100.0	136
cr40x5b-3	5279	8	3	14	102.7	100.9	97.3	98.6	5.6	97.1	5.1	99.4	3.0	1610	955	45	100.0	33

Table 10: Results on Class B: Prins et al. (2004) LRP based instances

Name	z^*	$\#r$	$\#f$	$\#c$	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_C$	t_C	$\%LB$	t_{LB}	$\#cuts$	$\#cols$	#N	$\%Opt$	t_{TOT}
ppw-20-5-0-a	1535	5	2	7	101.2	100.0	95.3	97.0	0.8	95.6	1.4	99.4	0.4	29	342	7	100.0	4
ppw-20-5-0-b	1315	3	1	8	103.3	100.0	93.0	94.4	1.3	99.2	0.6	99.6	7.6	632	3212	5	100.0	13
ppw-20-5-2-a	1488	5	3	11	102.1	100.0	94.8	95.6	2.1	92.4	1.4	97.3	0.4	300	324	133	100.0	11
ppw-20-5-2-b	1085	3	2	11	103.0	100.0	94.7	99.2	1.9	100.0	0.4	100.0	0.2	1	539	1	100.0	4
ppw-50-5-0-a	4159	12	3	12	102.3	100.7	97.5	99.3	1.7	93.5	15.7	99.3	0.9	7	650	645	100.0	96
ppw-50-5-0-b	2638	6	3	15	100.0	104.2	91.0	96.5	17.6	97.0	13.1	97.1	8.5	981	2751	283	97.3	3633
ppw-50-5-2'-a	4522	12	2	9	100.0	100.5	95.2	95.6	19.3	94.7	16.4	98.1	2.0	316	849	1057	100.0	214
ppw-50-5-2'-b	2439	6	1	7	100.9	100.3	90.8	92.5	7.7	97.7	12.1	98.9	9.8	630	2811	847	99.1	3633
ppw-50-5-2-a	3910	12	2	8	101.8	101.0	97.0	98.3	18.5	92.8	11.3	98.3	0.7	342	719	8849	100.0	946
ppw-50-5-2-b	2389	6	2	10	101.1	100.0	90.7	91.8	7.6	93.6	11.2	94.9	12.8	1414	2733	60	95.2	3623
ppw-50-5-3-a	3649	12	4	14	103.8	101.3	95.3	95.7	39.6	93.7	12.8	95.9	0.8	2	673	19638	98.3	3668
ppw-50-5-3-b	2421	6	3	25	100.0	100.6	90.0	93.8	4.1	92.9	5.7	95.7	9.6	1596	3006	199	96.2	3620
ppw-100-5-0-a	8009	25	3	10	101.5	101.1	97.2	98.5	3.1	93.3	210.1	98.7	7.2	11	1262	12954	99.6	3689
ppw-100-5-0-b	4629	11	4	13	100.0	103.7	93.0	95.7	219.0	92.4	116.9	97.3	50.9	3561	4394	38	97.4	3891
ppw-100-5-2-a	6838	24	3	13	104.1	100.0	96.6	97.2	105.8	91.6	169.9	97.3	6.5	645	1672	8072	97.7	3838
ppw-100-5-2-b	3925	11	3	20	100.4	100.0	93.6	94.8	110.1	89.8	133.6	94.9	17.5	657	2893	326	95.1	3793
ppw-100-5-3-a	7184	24	4	15	103.7	102.7	97.0	97.9	18.2	92.0	124.9	98.3	10.2	1291	1617	5523	98.7	3785
ppw-100-5-3-b	4141	11	4	21	100.0	101.5	93.3	95.3	27.3	92.0	133.4	96.1	44.6	3258	5104	50	96.3	3706
ppw-100-10-0-a	7960	24	8	28	101.2	100.0	95.5	96.8	166.0	88.3	109.0	96.8	5.9	2	1030	6269	97.4	3824
ppw-100-10-0-b	4698	26	6	35	100.0	103.6	93.4	95.3	61.8	91.7	164.9	95.6	24.0	274	3132	55	95.8	3721
ppw-100-10-2-a	6883	23	6	24	101.8	100.0	94.9	96.4	115.1	90.8	178.6	96.4	6.5	1	1988	2439	96.6	3831
ppw-100-10-2-b	3984	11	5	22	100.0	100.8	92.5	94.4	95.3	90.1	120.4	94.5	35.3	37	2982	71	94.6	3780
ppw-100-10-3-a	7060	24	9	32	102.3	100.0	95.6	96.4	147.3	87.4	124.5	97.1	10.5	5	1191	3973	97.5	3802
ppw-100-10-3-b	4081	11	6	22	100.0	102.0	92.2	94.1	75.2	90.5	138.9	95.2	68.3	1697	5956	92	95.3	3750

Table 11: Results on Class B: different authors LRP based instances

Name	z^*	$\#r$	$\#f$	$\#c$	$\%UB_1$	$\%UB_2$	$\%LB_1$	$\%LB_2$	t_{DA}	$\%LB_C$	t_C	$\%LB$	t_{LB}	$\#cuts$	$\#cols$	#N	$\%Opt$	t_{TOT}
Christ-50x5	3157	5	4	24	100.0	105.3	97.5	98.8	7.0	99.1	10.4	99.6	8.8	1410	2341	27	100.0	125
Christ-50x5_B	3238	5	5	21	103.9	103.0	96.5	98.5	6.7	99.0	8.8	99.3	11.1	1137	2467	63	99.7	3619
Christ-75x10	5241	9	5	17	100.0	101.4	92.1	92.9	60.4	90.8	45.9	94.4	42.1	1498	4130	83	94.6	3697
Christ-75x10_B	5394	9	5	18	100.0	102.5	93.5	94.6	50.8	92.2	37.8	96.0	55.8	1375	4226	3	96.0	3687
Christ-100x10	5377	8	5	34	100.0	103.9	92.3	93.8	70.2	93.3	63.8	94.9	91.9	813	9094	19	94.9	3735
Gaskell-21x5	2057	4	3	14	100.6	104.6	96.8	98.4	1.8	98.3	1.0	99.4	0.4	285	555	5	100.0	5
Gaskell-22x5	3097	3	3	14	105.5	100.0	78.4	75.9	86.7	96.4	0.8	99.6	147.4	560	3709	7	100.0	268
Gaskell-29x5	3035	3	2	5	103.9	105.0	91.4	96.2	103.1	93.7	1.6	99.4	177.0	69	3273	24	99.4	3708
Gaskell-32x5-2	2537	3	1	8	101.7	101.7	89.1	97.2	488.0	98.8	2.1	99.4	262.4	305	2208	39	100.0	1320
Gaskell-32x5	2691	4	2	9	105.1	100.0	97.1	99.2	206.7	99.3	1.7	99.5	55.2	24	2038	7	100.0	314
Gaskell-36x5	2355	4	3	24	109.8	100.0	98.7	98.8	1.3	99.8	3.8	100.0	6.8	1699	2661	5	100.0	19
Min-27x5	20229	4	2	10	103.9	102.2	82.6	93.8	4.0	95.1	1.4	96.5	4.9	368	1738	100	96.9	3848
Perl83-12x2	519	2	2	12	101.2	100.0	99.3	99.4	0.5	100.0	1.0	100.0	0.5	32	151	1	100.0	1
Perl83-55x15	2404	10	7	38	104.1	104.0	98.5	99.0	10.5	98.4	24.7	99.5	3.6	157	1867	137	100.0	221
Perl83-85x7	3994	11	6	33	101.3	101.1	97.1	97.9	44.4	95.3	62.8	98.0	17.1	1136	4049	247	98.3	3745
P111112-100x10	8778	11	6	36	100.0	102.3	93.4	95.2	58.8	92.2	90.8	95.8	43.9	325	5618	47	95.9	3737
P111122-100x20	8364	11	8	34	100.0	103.3	91.5	93.3	89.6	91.5	180.7	95.4	93.8	192	5649	26	95.4	3773
P111212-100x10	8371	10	8	47	100.9	100.0	92.4	94.8	33.2	93.4	163.3	95.2	57.4	2925	5442	21	95.3	3704
P111222-100x20	8732	11	8	44	100.0	105.3	91.2	92.8	43.0	92.2	227.5	95.5	121.6	442	8533	26	95.6	3718
P112112-100x10	8367	11	4	10	100.0	100.4	87.9	91.9	283.2	91.0	155.2	94.4	207.0	4431	8546	15	94.5	3976
P112122-100x20	6856	10	4	14	100.0	102.2	87.7	88.9	213.6	90.5	219.2	94.2	361.7	2199	14341	2	94.2	3905
P112212-100x10	6024	11	2	16	100.9	100.0	91.0	93.3	285.9	91.3	124.7	94.3	52.7	613	5383	274	94.4	3983
P112222-100x20	6869	10	3	15	100.0	103.0	94.4	95.3	91.1	95.0	221.4	96.5	301.9	5200	8629	28	96.6	3743
P113112-100x10	7987	10	3	11	100.6	100.0	86.8	89.0	276.6	87.0	116.8	90.3	150.0	4002	5866	18	90.5	3916
P113122-100x20	7573	11	4	18	100.0	105.4	92.3	93.9	90.0	93.5	176.8	96.7	624.9	8229	13890	13	96.8	3774
P113212-100x10	7888	10	2	6	100.0	106.0	94.9	95.6	258.7	94.6	153.7	97.1	62.6	725	5611	218	97.2	3925
P113222-100x20	7418	10	5	16	100.0	103.5	91.6	93.0	61.9	93.6	260.8	95.0	252.4	1191	8197	20	95.1	3733
P131112-150x10	12681	15	8	43	100.6	100.0	91.7	93.1	119.3	88.9	686.9	93.3	74.6	55	8146	61	93.4	3896
P131122-150x20	11881	16	14	68	100.0	102.6	91.2	93.2	163.5	90.0	917.0	94.2	175.2	225	9803	88	94.3	3978
P131212-150x10	12314	16	5	21	100.0	102.1	93.0	95.8	427.6	90.7	443.6	96.3	132.1	441	10244	24	96.4	4256
P131222-150x20	11858	15	10	74	100.0	102.1	93.6	94.7	159.9	89.3	748.6	95.1	118.1	459	7479	125	95.2	3906
P132112-150x10	11952	16	3	14	100.0	103.5	93.9	95.7	629.4	92.5	879.6	96.1	81.3	358	7255	22	96.2	4478
P132122-150x20	10198	15	7	37	100.0	101.3	90.5	91.3	569.3	90.8	1193.7	93.1	221.5	795	14115	42	93.1	4343
P132212-150x10	8683	16	1	9	100.0	101.1	91.9	93.0	577.9	90.8	893.1	93.7	243.7	2658	9313	25	93.7	4411
P132222-150x20	8218	16	3	20	100.0	102.2	92.8	94.0	543.6	91.1	955.8	94.3	201.0	2589	8535	37	94.3	4397
P133112-150x10	13043	16	5	11	100.0	105.3	91.7	93.3	152.4	92.0	1048.9	94.6	496.3	6615	11122	8	94.6	3970
P133122-150x20	10027	16	6	24	100.0	104.5	92.9	94.0	464.2	91.0	886.2	94.9	149.8	1643	8831	116	95.0	4300
P133212-150x10	8707	16	1	7	100.0	104.0	92.7	93.9	225.6	92.5	778.3	95.2	95.7	1548	7539	189	95.3	4088
P133222-150x20	10822	16	4	11	100.0	103.0	89.9	90.8	355.6	88.2	849.5	91.1	303.4	2350	9247	30	91.2	4209