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VOLUME OF SUBLEVEL SETS OF HOMOGENEOUS POLYNOMIALS

JEAN B. LASSERRE

ABSTRACT. Consider the compact sub-level set $\mathbf{K} := \{\mathbf{x} : g(\mathbf{x}) \leq 1\}$ of a nonnegative homogeneous polynomial g. We show that its Lebesgue volume $\mathrm{vol}(\mathbf{K})$ can be approximated as closely as desired by solving a sequence of generalized eigenvalue problems with respect to a pair of Hankel matrices of increasing size, whose entries are obtained in closed form. The methodology also extends to compact sets of the form $\{\mathbf{x} : a \leq g(\mathbf{x}) \leq b\}$ for non-homogeneous polynomials with degree $d \ll n$. It reduces the volume computation in $\mathbb{R}^n$ to a "volume" computation in $\mathbb{R}^d$ (where $d = \deg(g)$) for a certain pushforward measure. Another extension to computing volumes of finite intersections of such sub-level sets is also briefly described.

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1. INTRODUCTION

Let $g \in \mathbb{R}[\mathbf{x}]_t$ be a nonnegative homogeneous polynomial of degree t (hence t is even) with associated compact sub-level set

$$\mathbf{K} := \{\, \mathbf{x} \in \mathbb{R}^n : \, g(\mathbf{x}) \leq 1 \,\}, \tag{1.1}$$

and with no loss of generality we may and will assume that (possibly after scaling) $\mathbf{K} \subset \mathbf{B}$ where $\mathbf{B}$ is the unit box $[-1, 1]^n$. In this paper we describe an efficient numerical scheme to approximate its Lebesgue volume $\mathrm{vol}(\mathbf{K})$ (when finite) as closely as desired.

Motivation. In addition of being an interesting mathematical problem on its own, computing $\mathrm{vol}(\mathbf{K})$ has also a practical interest outside computational geometry. For instance it has a direct link with computing the integral $\int \exp(-g(\mathbf{x}))d\mathbf{x}$, called an *integral discriminant* in Dolotin and Morozov [2] and Morozov and Shakirov [11]. Indeed as proved in [11]:

$$\mathrm{vol}(\mathbf{K}) \;=\; \frac{1}{\Gamma(1 + \frac{n+t}{2})} \int_{\mathbb{R}^n} \exp(-g(\mathbf{x}))\, d\mathbf{x}, \tag{1.2}$$

and to quote [11], *"averaging with exponential weights is an important operation in statistical and quantum physics"*. However, and again quoting [11], *"despite simply looking,* (1.2) *remains terra incognita"*. Nevertheless, for special cases of homogeneous polynomials, the authors in [11] have been able to obtain a closed form

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expression for (1.2) (hence equivalently for vol($\mathbf{K}$)) in terms of algebraic invariants of g.

Various consequences of formula (1.2) have been described and exploited in Lasserre [6]. For instance, vol($\mathbf{K}$) is a convex function in the coefficients of the polynomial g. In particular this strong property has been exploited for proving an extension of the Löwner-John ellipsoid theorem [7] which permits to completely characterize the sublevel set $\mathbf{K}$ (as in (1.1)) of minimum volume which contains a given set $\mathbf{\Omega} \subset \mathbb{R}^n$ (when minimizing over all positive homogeneous polynomials g of degree t). But computing this sublevel set of minimum volume that contains $\mathbf{K}$ is a computational challenge since computing (or even approximating) the integral (1.2) is a hard problem. Our main result is that

$$\text{vol}(\mathbf{K}) \;=\; \lim_{d\to\infty} \lambda_{\min}(\mathbf{A}_d, \mathbf{B}_d), \tag{1.3}$$

where $\lambda_{\min}(\mathbf{A}_d, \mathbf{B}_d)$ is the smallest generalized eigenvalue of the pair $(\mathbf{A}_d, \mathbf{B}_d)$. More precisely, vol($\mathbf{K}$) (and therefore the integral discriminant (1.2)) is the limit of a monotone sequence of generalized eigenvalue problems with respect to a pair $(\mathbf{A}_d, \mathbf{B}_d)$ of given real Hankel matrices of size $d+1$. All entries of both Hankel matrices are easy to obtain in form and the Hankel matrix $\mathbf{B}_d$ depends only on the degree of g. Therefore, in principle the integral (1.2) can be approximated efficiently and as closely as desired by (linear algebra) eigenvalue routines. To the best of our knowledge this result is quite new and in addition, even if we do not provide a closed form expression of (1.2), its new characterization as a limit or eigenvalue problems could bring new insights. Moreover, a first set of numerical experiments on an academic problem (retrieving the volume of the Euclidean unit ball in $\mathbb{R}^n$) to verify the behavior of $\lambda_{\min}(\mathbf{A}_d, \mathbf{B}_d)$ as d increases, shows a quick convergence with quite precise approximations obtained with relatively small d; for instance, with $d = 8$, the relative error is only 0.6% for $n = 8$ and 2.15% for $n = 9$.

Methodology. Computing (and even approximating) the Lebesgue volume of a convex body is hard (let alone non-convex bodies). Often the only possibility is to use (non deterministic) Monte Carlo type methods which provide an estimate with statistical guarantees; that is, generate a sample of N points according to the uniform distribution on $[-1,1]^n$ and then the ratio $\rho_N :=$ (number of points in $\mathbf{K}$)$/N$ provides such an estimate. However ρ_N is a random variable and is neither an upper bound or a lower bound on vol($\mathbf{K}$). For a discussion on volume computation the interested reader is referred to [4] and the many references therein.

However for basic semi-algebraic sets $\mathbf{K} \subset [-1,1]^n$, Henrion et al. [4] have provided a general methodology to approximate vol($\mathbf{K}$). It consists in solving a hierarchy $(\mathbf{Q}_d)_{d\in\mathbb{N}}$ of semidefinite programs of increasing size, whose associated sequence of optimal values $(\rho_d)_{d\in\mathbb{N}}$ is monotone non increasing and converges to vol($\mathbf{K}$). A semidefinite program (SDP) is a conic convex optimization problem with a remarkable modeling power. It can be solved efficiently in time polynomial in its input size, up to arbitrary precision fixed in advance; for more details the interested reader is referred to e.g. Anjos and Lasserre [1].

An optimal solution of $\mathbf{Q}_d$ is a vector $\mathbf{y} \in \mathbb{R}^{s(2d)}$ (with $s(d) = \binom{n+d}{n}$) whose each coordinate y_α, $\alpha \in \mathbb{N}^n_{2d}$, approximates the α-moment of $\lambda_{\mathbf{K}}$, the restriction to $\mathbf{K}$ of the Lebesgue measure λ on $\mathbb{R}^n$; therefore y_0 approximates vol($\mathbf{K}$) from above. An optimal solution of the dual semidefinite program $\mathbf{Q}^*_d$ provides the coefficients $(p_\alpha)_{\alpha\in\mathbb{N}^n_{2d}}$ of a polynomial $p \in \mathbb{R}[\mathbf{x}]_{2d}$ which approximates on $[-1,1]$

and from above, the (indicator) function $\mathbf{x} \mapsto 1_{\mathbf{K}}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbf{K}$ and 0 otherwise. In general the convergence $\rho_d \to \mathrm{vol}(\mathbf{K})$ is slow because of a Gibbs phenomenon[1] when one approximates the indicator function $1_{\mathbf{K}}$ by continuous functions. In [4] the authors have proposed a "tric" which accelerate drastically the convergence but at the price of loosing the monotone convergence $\rho_d \downarrow \mathrm{vol}(\mathbf{K})$. Another acceleration technique was provided in [8] which still preserves monotone convergence. It uses the fact that moments of $\lambda_{\mathbf{K}}$ satisfy linear equality constraints that follow from Stokes' theorem.

Recently, Jasour et al. [5] have considered volume computation in the context of risk estimation in uncertain environments. They have provided an elegant "trick" which reduces computing the n-dimensional volume $\mathrm{vol}(\mathbf{K})$ to computing $\phi([0,1])$ for a certain pushforward measure ϕ on the real line, whose moments are known. With $\mathbf{K}$ as in (1.1) the pushforward measure ϕ is with respect to the mapping g. This results in solving the hierarchy of semidefinite programs proposed in [4], but now for measures on the real line as opposed to measures on $\mathbb{R}^n$. Solving the corresponding hierarchy of dual semidefinite programs amounts to approximate the indicator of an interval on the real line by polynomials of increasing degree, and whose coefficients minimize a linear criterion.

On the one hand, it yields drastic computational savings as passing from $\mathbb{R}^n$ to $\mathbb{R}$ is indeed a big and impressive progress. But on the other hand the monotone convergence remains slow as one cannot apply the acceleration technique based on Stokes' theorem proposed e.g. in [8] because the density of ϕ is not known explicitly. In the examples provided in §3.2 for comparison, we can observe this typical very slow convergence. However as the problem is now one-dimensional one may then solve many more steps of the resulting hierarchy of semidefinite programs provided that one works with a nice basis of polynomials, e.g., Chebyshev polynomials, to avoid numerical problems as much as possible. Interestingly, pushforward measures were also used in Magron et al. [10] to compute the Lebesgue volume of $f(\mathbf{K})$ for a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^m$. However in this case one has to compute moments of the measure in $\mathbb{R}^n$ whose pushforward measure is the Lebesgue measure on $f(\mathbf{K})$. The resulting computation is still very expensive and limited to modest dimensions.

Contribution. We provide a simple numerical scheme to approximate $\mathrm{vol}(\mathbf{K})$ with $\mathbf{K}$ as in (1.1) and when g is positive and homogeneous. To do so we are inspired by the trick of using the pushforward measure in Jasour et al. [5]. The novelty here is that by taking into account the specific nature (homogeneity) of g in (1.1), we are able to drastically simplify computations. Indeed the hierarchy of semidefinite programs defined in [5] can be replaced and significantly improved, with computing a sequence of scalars $(\tau_d)_{d\in\mathbb{N}}$. Each τ_d is nothing less than the generalized minimum eigenvalue of two known Hankel matrices of size d, whose entries are obtained exactly in closed form with no numerical error. Therefore there is *no* optimization involved any more. Moreover, if one uses the basis of orthonormal polynomials w.r.t. the pushforward measure, then τ_d is now the minimum eigenvalue of a single real symmetric matrix of size d.

At last but not least, the philosophy underlying the methodology also extends to arbitrary compact sets of the form $\{\mathbf{x} : a \le g(\mathbf{x}) \le b\} \subset \mathbb{R}^n$ where the polynomial

[1]The Gibbs' phenomenon appears at a jump discontinuity when one approximates a piecewise C^1 function with a continuous function, e.g. by its Fourier series.

g is not necessarily homogeneous and positive. This can be potentially interesting when $\deg(g) \ll n$ because we reduce the initial Lebesgue volume computation in $\mathbb{R}^n$ to a μ-volume computation in $\mathbb{R}^t$, where $t = \deg(g)$. The measure μ on $\mathbb{R}^t$ is a certain pushforward measure whose sequence of moments is easily obtained in closed form. Moreover, and crucial for the approach, we are still able to include additional constraints based on Stokes' theorem, which significantly accelerates the otherwise typically slow convergence. In [5] the problem would be reduced to a ν-volume computation only in $\mathbb{R}$ but with no possibility to include additional Stokes' constraints to accelerate the slow convergence.

The paper is organized as follows. After some preliminaries in §2, our main results as well as some experimental results are found in §3 and some extensions in §4.

2. Notation and definitions

Notation. Let $\mathbb{R}[\mathbf{x}]$ denote the ring of polynomials in the variables $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbb{R}[\mathbf{x}]_t \subset \mathbb{R}[\mathbf{x}]$ denote the vector space of polynomials of degree at most t, hence of dimension $s(t) = \binom{n+t}{n}$. Let $\Sigma[\mathbf{x}] \subset \mathbb{R}[\mathbf{x}]$ denote the space of polynomials that are sums-of-squares (in short SOS polynomials) and let $\Sigma[\mathbf{x}]_t \subset \mathbb{R}[\mathbf{x}]_{2t}$ denote the space of SOS polynomials of degree at most $2t$. With $\alpha \in \mathbb{N}^n$ and $\mathbf{x} \in \mathbb{R}^n$, the notation $\mathbf{x}^\alpha$ stands for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Also for every $\alpha \in \mathbb{N}^n$, let $|\alpha| := \sum_i \alpha_i$ and $\mathbb{N}^n_t := \{\alpha \in \mathbb{N}^n : |\alpha| \leq t\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The support of a Borel measure μ on $\mathbb{R}^n$ is the smallest closed set $\mathbf{\Omega}$ such that $\mu(\mathbb{R}^n \setminus \mathbf{\Omega}) = 0$. Denote by $\mathcal{B}(\mathbf{X})$ the Borel σ-field associated with a topological space $\mathbf{X}$, and $\mathscr{M}(\mathbf{X})$ the space of finite nonnegative Borel measures on $\mathbf{X}$.

Generalized eigenvalue. Given two real symmetric matrices $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{n\times n}$ denote by $\lambda_{\min}(\mathbf{A}, \mathbf{C})$ the smallest generalized eigenvalue with respect to the pair $(\mathbf{A}, \mathbf{C})$, that is, the smallest scalar τ such that $\mathbf{A}\mathbf{x} = \tau\, \mathbf{C}\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$. When $\mathbf{C}$ is the identity matrix then $\lambda_{\min}(\mathbf{A}, \mathbf{C})$ is just the smallest eigenvalue of $\mathbf{A}$. Computing $\lambda_{\min}(\mathbf{A}, \mathbf{C})$ can be done via a pure and efficient linear algebra routine. The notation $\mathbf{A} \succeq 0$ (resp. $\mathbf{A} \succ 0$) stands for $\mathbf{A}$ is positive semidefinite (resp. positive definite). If $\mathbf{C} \succ 0$ then:

$$\lambda_{\min}(\mathbf{A}, \mathbf{C}) \;=\; \max_{\tau}\{\, \tau : \mathbf{A} \succeq \tau\, \mathbf{C}\,\}. \tag{2.1}$$

Moment matrix. Given a real sequence $\boldsymbol{\phi} = (\phi_\alpha)_{\alpha\in\mathbb{N}^n}$, let $\mathbf{M}_d(\boldsymbol{\phi})$ denote the multivariate (Hankel-type) moment matrix defined by $\mathbf{M}_d(\boldsymbol{\phi})(\alpha, \beta) = \phi_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{N}^n_d$. For instance, in the univariate case $n = 1$, with $d = 2$, $\mathbf{M}_2$ is the Hankel matrix

$$\mathbf{M}_2(\boldsymbol{\phi}) \;=\; \begin{bmatrix} \phi_0 & \phi_1 & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \\ \phi_2 & \phi_3 & \phi_4 \end{bmatrix}.$$

If $\boldsymbol{\phi} = (\phi_j)_{j\in\mathbb{N}}$ is the moment sequence of a Borel measure ϕ on $\mathbb{R}$ then $\mathbf{M}_d(\boldsymbol{\phi}) \succeq 0$ for all $d = 0, 1, \ldots$. Conversely, if $\mathbf{M}_d(\boldsymbol{\phi}) \succeq 0$ for all $d \in \mathbb{N}$, then $\boldsymbol{\phi}$ is the moment sequence of some finite (nonnegative) Borel measure ϕ on $\mathbb{R}$. The converse result is not true anymore in the multivariate case.

Let ϕ, ν be two finite Borel measures on $\mathbb{R}$. The notation $\phi \leq \nu$ stands for $\phi(B) \leq \nu(B)$ for all $B \in \mathcal{B}(\mathbb{R})$.

Lemma 2.1. *Let ϕ, ν be two finite nonnegative Borel measures on $\mathbb{R}$ with all moments $\boldsymbol{\phi} = (\phi_j)_{j\in\mathbb{N}}$ and $\boldsymbol{\nu} = (\nu_j)_{j\in\mathbb{N}}$ finite. Then $\phi \le \nu$ if and only if*

$$\mathbf{M}_d(\boldsymbol{\phi}) \preceq \mathbf{M}_d(\boldsymbol{\nu}), \quad \forall d = 0, 1, \ldots$$

Proof. Only if part: $\phi \le \nu$ implies that $\nu - \phi$ with associated sequence $\boldsymbol{\nu} - \boldsymbol{\phi} = (\nu_j - \phi_j)_{j\in\mathbb{N}}$ is a finite nonnegative Borel measure on $\mathbb{R}$, and therefore:

$$\mathbf{M}_d(\boldsymbol{\nu}) - \mathbf{M}_d(\boldsymbol{\phi}) = \mathbf{M}_d(\boldsymbol{\nu} - \boldsymbol{\phi}) \succeq 0, \quad d \in \mathbb{N},$$

i.e., $\mathbf{M}_d(\boldsymbol{\nu}) \succeq \mathbf{M}_d(\boldsymbol{\phi})$ for all $d \in \mathbb{N}$.

If part: If $\mathbf{M}_d(\boldsymbol{\phi}) \preceq \mathbf{M}_d(\boldsymbol{\nu})$ for all $d \in \mathbb{N}$ then the sequence $\boldsymbol{\nu} - \boldsymbol{\phi} = (\nu_j - \phi_j)_{j\in\mathbb{N}}$ satisfies $\mathbf{M}_d(\boldsymbol{\phi} - \boldsymbol{\nu}) \succeq 0$ for all $d \in \mathbb{N}$. Therefore, the moment sequence $\boldsymbol{\nu} - \boldsymbol{\phi} = (\nu_j - \phi_j)_{j\in\mathbb{N}}$ of the possibly signed measure $\nu - \phi$ is in fact the moment sequence of a finite nonnegative Borel measure on $\mathbb{R}$, and therefore $\nu \ge \phi$. □

Localizing matrix. Given a real sequence $\boldsymbol{\phi} = (\phi_\alpha)_{\alpha\in\mathbb{N}^n}$ and a polynomial $\mathbf{x} \mapsto p(\mathbf{x}) := \sum_\gamma p_\gamma \mathbf{x}^\gamma$, let $\mathbf{M}_d(p\,\boldsymbol{\phi})$ denote the real symmetric matrix defined by:

$$\mathbf{M}_d(p\,\boldsymbol{\phi})(\alpha, \beta) = \sum_\gamma p_\gamma\, \phi_{\alpha+\beta+\gamma}, \quad \alpha, \beta \in \mathbb{N}^n_d.$$

For instance, with $n = 1$, $d = 2$ and $x \mapsto p(x) = x(1 - x)$:

$$\mathbf{M}_2(p\,\boldsymbol{\phi}) = \begin{bmatrix} \phi_1 - \phi_2 & \phi_2 - \phi_3 & \phi_3 - \phi_4 \\ \phi_2 - \phi_3 & \phi_3 - \phi_4 & \phi_4 - \phi_5 \\ \phi_3 - \phi_4 & \phi_4 - \phi_5 & \phi_5 - \phi_6 \end{bmatrix},$$

also a Hankel matrix.

Lemma 2.2. *Let $x \mapsto p(x) = x(1 - x)$.*

(i) If a real (finite) sequence $\boldsymbol{\phi} = (\phi_j)_{j\le 2d}$ satisfies $\mathbf{M}_d(\boldsymbol{\phi}) \succeq 0$ and $\mathbf{M}_{d-1}(p\,\boldsymbol{\phi}) \succeq 0$, then there is a nonnegative Borel measure μ on $[0, 1]$ whose moments $\boldsymbol{\mu} = (\mu_j)_{j\le 2d}$ match $\boldsymbol{\phi}$.

(ii) If a real (infinite) sequence $\boldsymbol{\phi} = (\phi_j)_{j\in\mathbb{N}}$ satisfies $\mathbf{M}_d(\boldsymbol{\phi}) \succeq 0$ and $\mathbf{M}_d(p\,\boldsymbol{\phi}) \succeq 0$ for all d, then there is a nonnegative Borel measure μ on $[0, 1]$ whose moments $\boldsymbol{\mu} = (\mu_j)_{j\in\mathbb{N}}$ match $\boldsymbol{\phi}$.

See for instance Lasserre [9] and the many references therein.

Pushforward measure. Let $\mathbf{K} \subset \mathbb{R}^n$ be a Borel set and λ a probability measure on $\mathbf{K}$. Given a measurable mapping $f : \mathbf{K} \to \mathbb{R}^p$, the pushforward (nonnegative) Borel measure of λ on $\mathbb{R}^p$ w.r.t. f, is denoted by $\#\lambda$ and satisfies:

$$\#\lambda(B) := \lambda(f^{-1}(B)), \qquad \forall B \in \mathcal{B}(\mathbb{R}^p).$$

In particular, its moments are given by

$$\#\lambda_\alpha := \int_{\mathbb{R}^p} \mathbf{z}^\alpha \, \#\lambda(d\mathbf{z}) = \int_{\mathbf{K}} f(\mathbf{x})^\alpha \, \lambda(d\mathbf{x}), \qquad \alpha \in \mathbb{N}^p. \tag{2.2}$$

A version of Stokes' theorem. Let $\boldsymbol{\Omega} \subset \mathbb{R}^n$ be a smooth manifold with boundary $\partial\boldsymbol{\Omega}$ and let $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x})$ be a given vector field. Let "Div" denote the Divergence operator so that $\mathrm{Div}(\mathbf{X}) = \sum_i \partial\mathbf{X}/\partial x_i$. Then Stokes' theorem states that

$$\int_{\boldsymbol{\Omega}} \mathrm{Div}(\mathbf{X}) f(\mathbf{x})\, d\mathbf{x} + \int_{\boldsymbol{\Omega}} \langle \mathbf{X}, \nabla f(\mathbf{x}) \rangle \, d\mathbf{x} = \int_{\partial\boldsymbol{\Omega}} \langle \vec{n}_\mathbf{x}, \mathbf{X} \rangle f(\mathbf{x})\, d\sigma, \tag{2.3}$$

where $\vec{n}_{\mathbf{x}}$ is the outward pointing normal to $\mathbf{\Omega}$ at $\mathbf{x} \in \partial\mathbf{\Omega}$, and σ is the $(n-1)$-dimensional Hausdorff measure on the boundary $\partial\mathbf{\Omega}$; see e.g. Taylor [12, Proposition 3.2, p. 128]. Then Whitney [13, Theorem 14A] generalized Stokes' theorem to rough domains $\mathbf{\Omega}$, e.g. domains with corners.

3. Main result

Let $\mathbf{K}$ in (1.1) be strictly contained in $\mathbf{B} := [-1,1]^n$, and with boundary $\partial\mathbf{K} = \{\mathbf{x} : g(\mathbf{x}) = 1\}$. Let λ be the Lebesgue measure on $\mathbf{B}$ normalized to a probability measure so that $\mathrm{vol}(\mathbf{K}) = 2^n\lambda(\mathbf{K})$. Let g in (1.1) be a nonnegative and homogeneous polynomial of degree t. That is, $g(\theta\mathbf{x}) = \theta^t g(\mathbf{x})$ for all $\theta \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$. Denote by:

$$\overline{b}_g := \max\{\, g(\mathbf{x}) : \mathbf{x} \in \mathbf{B}\,\}; \quad \underline{a}_g := \min\{\, g(\mathbf{x}) : \mathbf{x} \in \mathbf{B}\,\}. \tag{3.1}$$

Notice that $\underline{a}_g = 0$ because $0 \in \mathbf{B}$ and g is nonnegative with $g(0) = 0$, and therefore $g(\mathbf{B}) = [0, \overline{b}_g]$. We next follow an elegant idea of Jasour et al. [5] adapted to the present context. It reduces the computation of $\mathrm{vol}(\mathbf{K})$ in $\mathbb{R}^n$ to a certain volume computation in $\mathbb{R}$, by using a particular pushforward measure of the Lebesgue measure λ on $\mathbf{B}$.

Let $\#\lambda$ be a measure on the positive half line, the pushforward measure of λ by the polynomial mapping $g : \mathbf{B} \to [0, \overline{b}_g]$. From (3.1), the support of $\#\lambda$ is the interval $I := [0, \overline{b}_g] \subset \mathbb{R}$. Then in view of (2.2):

$$\#\lambda_k := \int_I z^k \,\#\lambda(dz) = \int_{\mathbf{B}} g(\mathbf{x})^k \,\lambda(d\mathbf{x}), \quad k = 0, 1, \ldots \tag{3.2}$$

All scalars $(\#\lambda_k)_{k\in\mathbb{N}}$ can be obtained in closed form as g is a polynomial and λ is the (normalized) Lebesgue measure on $\mathbf{B}$. Namely, writing the expansion

$$\mathbf{x} \mapsto g(\mathbf{x})^k = \sum_{\alpha\in\mathbb{N}^n_{kd}} g_{k\alpha}\,\mathbf{x}^\alpha,$$

for some coefficients $(g_{k\alpha})$, one obtains:

$$\#\lambda_k = 2^{-n} \sum_{\alpha\in\mathbb{N}^n_{kd}} g_{k\alpha} \left(\prod_{i=1}^n \frac{(1-(-1)^{\alpha_i+1})}{\alpha_i+1} \right), \quad k = 0, 1, \ldots \tag{3.3}$$

Next observe that $2^{-n}\mathrm{vol}(\mathbf{K}) = \#\lambda(g(\mathbf{K}))$ and note that $g(\mathbf{K}) = [0,1]$. Therefore following the recipe introduced in Henrion et al. [4], and with $S := [0,1] \subset [0, \overline{b}_g]$:

$$\#\lambda(S) = \max_{\phi\in\mathscr{M}(S)} \{\, \phi(S) :\, \phi \le \#\lambda\,\}. \tag{3.4}$$

Denote by ϕ^* the nonnegative Borel measure on the real line which is the restriction to $S \subset I$ of the pushforward measure $\#\lambda$. That is,

$$\phi^*(B) := \#\lambda(B \cap S), \qquad \forall B \in \mathcal{B}(\mathbb{R}). \tag{3.5}$$

Then ϕ^* is the unique optimal solution of (3.4) and therefore $\phi^*(S) = \#\lambda(S)$; see e.g. Henrion et al. [4]. Then to approximate $\phi^*(S)$ from above, one possibility is to solve the hierarchy of semidefinite relaxations indexed by $d \in \mathbb{N}$:

$$\rho_d = \max_{\boldsymbol{\phi}}\{\, \phi_0 :\, 0 \preceq \mathbf{M}_d(\boldsymbol{\phi}) \preceq \mathbf{M}_d(\#\lambda); \quad \mathbf{M}_{d-1}(x(1-x)\,\boldsymbol{\phi}) \succeq 0\,\}, \tag{3.6}$$

where $\boldsymbol{\phi} = (\phi_j)_{j\le 2d}$, $\mathbf{M}_d(\#\lambda)$ is the Hankel moment matrix (with moments up to order $2d$) associated with the pushforward measure $\#\lambda$, and $\mathbf{M}_d(\boldsymbol{\phi})$ (resp. $\mathbf{M}_{d-1}(z(1-z)\,\boldsymbol{\phi})$) is the Hankel moment (resp. localizing) matrix with moments up to order $2d$, associated with the sequence $\boldsymbol{\phi}$ and the polynomial $x \mapsto p(x) = x(1-x)$; see §2. Indeed (3.6) is a relaxation of (3.4) and the sequence $(\rho_d)_{d\in\mathbb{N}}$ is monotone non increasing and converges to $\phi^*(S) = \#\lambda(S)$ from above; see e.g. [4].

The dual of (3.6) is the semidefinite program

$$\rho_d^* = \max_{p\in\mathbb{R}[x]_{2d}} \{ \int p\, d\#\lambda : \quad p - 1 = \sigma_0 + \sigma_1\, x(1-x) \\ p, \sigma_0 \in \Sigma[x]_d; \quad \sigma_1 \in \Sigma[x]_{d-1} \}, \tag{3.7}$$

and if $\mathbf{K}$ has nonempty interior then $\rho_d^* = \rho_d$. This is the approach advocated by Jasour et al. [5] and indeed this reduction of the initial (Lebesgue) volume computation in $\mathbb{R}^n$

$$\mathrm{vol}(\mathbf{K}) \;=\; \max_{\phi\in\mathscr{M}(\mathbf{K})} \{\, \phi(\mathbf{K}) :\; \phi \;\le\; \lambda \,\} \tag{3.8}$$

to instead compute $\#\lambda([0,1])$ (in $\mathbb{R}$) by solving (3.4) is quite interesting as it yields drastic computational savings. In fact, solving the multivariate analogues for (3.8) of the univariate semidefinite relaxations (3.6) for (3.4), becomes rapidly impossible even for moderate d, except for problems of modest dimension (say e.g. $n \le 4$).

However it is important to notice that in general the convergence $\rho_d \downarrow \#\lambda(S)$ is very slow and numerical problems are expected even for not so large values of d. To partially remedy this problem the authors of [5] suggest to express moment and localizing matrices in (3.6) in the Chebyshev basis rather than in the standard monomial basis. However if this allows to solve a larger number of relaxations, it does not change the typical slow convergence. The trick based on Stokes' theorem used in [8] cannot be used here because the dominating (or reference) measure $\#\lambda$ in (3.4) is *not* the Lebesgue measure λ any more (as in (3.8)). On the other hand, the trick in [4] to accelerate convergence can still be used, that is, in (3.6) one now maximizes $L_{\boldsymbol{\phi}}(x(1-x)) = \phi_1 - \phi_2$ instead of ϕ_0. If $\boldsymbol{\phi}^d = (\phi_j^d)_{j\le 2d}$ is an optimal solution of (3.6) then $\phi_0^d \to \#\lambda(S)$ as d increases, but one looses the monotone convergence from above.

In the sequel we show that in the particular case where g is positive and homogeneous then one can avoid solving the hierarchy (3.6). Instead one solves a hierarchy of simple generalized eigenvalue problems with *no* optimization involved and with a much faster convergence.

3.1. **Exploiting homogeneity.**

A crucial observation. We apply Stokes' relation (2.3) with vector field $\mathbf{X}(\mathbf{x}) = \mathbf{x}$ to the function $1 - g(\mathbf{x})^j$ which vanishes on the boundary $\partial\mathbf{K}$.

Recall that $2^{-n}\mathrm{vol}(\mathbf{K}) = \#\lambda(S)$. So let ϕ^* be as in (3.5), and let $\boldsymbol{\phi}^* = (\phi_j^*)_{j\in\mathbb{N}}$ be its associated sequence of moments. Consider the vector field $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x}) := \mathbf{x}$. Then $\mathrm{Div}(\mathbf{X}) = \sum_i \partial X/\partial x_i = n$. In addition, as g is homogeneous of degree t then by Euler's identity, $\langle \mathbf{x}, \nabla g(\mathbf{x})\rangle = t\, g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Recall that $\mathbf{K} \subset \mathrm{int}(\mathbf{B})$ and therefore $g(\mathbf{x}) = 1$ for every $\mathbf{x} \in \partial\mathbf{K}$. Next, for every $j \in \mathbb{N}$, as $g(\mathbf{x})^j = 1$ on $\partial\mathbf{K}$

for all $j \in \mathbb{N}$, Stokes' Theorem (2.3) yields:

$$\begin{aligned} 0 &= \int_{\partial\mathbf{K}} \langle \vec{n}_{\mathbf{x}}, \mathbf{x}\rangle\,(1-g(\mathbf{x})^j)\,d\sigma \quad [\text{as } g(\mathbf{x})=1 \text{ on } \partial\mathbf{K}] \\ &= n\int_{\mathbf{K}} (1-g(\mathbf{x})^j)\,\lambda(d\mathbf{x}) + \int_{\mathbf{K}} \langle \mathbf{x}, \nabla(1-g(\mathbf{x})^j)\rangle\,\lambda(d\mathbf{x}) \quad [\text{by Stokes}] \\ &= n\,\lambda(\mathbf{K}) - (n+jt)\int_{\mathbf{K}} g(\mathbf{x})^j\,\lambda(d\mathbf{x}) \\ &= n\,\#\lambda(S) - (n+jt)\int_{g(\mathbf{K})} z^j\,\#\lambda(dz) \;=\; n\,\phi^*_0 - (n+jt)\,\phi^*_j, \end{aligned}$$

so that we have proved:

Lemma 3.1. *Let ϕ^* be the nonnegative Borel measure on $\mathbb{R}$ which is the restriction to $S=[0,1]$ of the pushforward measure $\#\lambda$ on I. Then its moments $\boldsymbol{\phi}^* = (\phi^*_j)_{j\in\mathbb{N}}$ satisfy :*

$$\phi^*_0 \;=\; 2^{-n}\,\mathrm{vol}(\mathbf{K}); \quad \phi^*_j \;:=\; \frac{n}{n+jt}\,\phi^*_0, \quad j=1,2,\ldots \tag{3.9}$$

Remarkably, (3.1) relates all moments of ϕ^* to its mass $\phi^*_0 = 2^{-n}\mathrm{vol}(\mathbf{K})$ in very simple manner! However it now remains to compute ϕ^*_0.

Computing ϕ^*_0. Define $\mathbf{M}^*_d$ to be the Hankel (moment) matrix with entries:

$$\mathbf{M}^*_d(k,\ell) \;:=\; \frac{n}{n+(k+\ell-2)\,t}, \quad k,\ell \;=\; 1,2,\ldots,d+1, \tag{3.10}$$

so that $\phi^*_0\,\mathbf{M}^*_d = \mathbf{M}_d(\boldsymbol{\phi}^*)$ for all $d\in\mathbb{N}$, where $\mathbf{M}_d(\boldsymbol{\phi}^*)$ is the Hankel moment matrix associated with the sequence $\boldsymbol{\phi}^*$ (equivalently with the measure ϕ^*).

Similarly, define $\mathbf{M}^*_{d,x(1-x)}$ to be the Hankel matrix with entries:

$$\mathbf{M}^*_{d,x(1-x)}(k,\ell) \;:=\; \frac{n}{n+(k+\ell-1)\,t} - \frac{n}{n+(k+\ell)\,t}, \quad k,\ell \;=\; 1,2,\ldots,d+1, \tag{3.11}$$

so that $\phi^*_0\,\mathbf{M}^*_{d,x(1-x)} = \mathbf{M}_d(x(1-x)\,\boldsymbol{\phi}^*)$ is the localizing matrix associated with $\boldsymbol{\phi}^*$ and the polynomial $x\mapsto x(1-x)$, for all $d\in\mathbb{N}$. As ϕ^* is supported on $[0,1]$ then $\phi^*_0\,\mathbf{M}^*_{d,x(1-x)} \succeq 0$ for all d, which in turn implies

$$\mathbf{M}^*_{d,x(1-x)} \;\succeq\; 0, \qquad \forall d\in\mathbb{N}, \tag{3.12}$$

because $\phi^*_0 > 0$.

Proposition 3.2. *The matrix $\mathbf{M}^*_d$ defined in* (3.10) *is the moment matrix (with moments up to order $2d$) of the nonnegative Borel measure $d\psi = \frac{n}{t}x^{\frac{n}{t}-1}dx$ on $[0,1]$. Therefore $\mathbf{M}^*_d \succ 0$ for all $d\in\mathbb{N}$.*

Proof. Indeed for every $j\in\mathbb{N}$:

$$\int_0^1 x^j\,d\psi(x) \;=\; \frac{n}{t}\int_0^1 x^j\,x^{\frac{n}{t}-1}\,dx \;=\; \frac{n}{n+jt}.$$

Finally, $\mathbf{M}^*_d \succ 0$ for all $d\in\mathbb{N}$ because the density $x^{\frac{n}{t}-1}$ is positive on $[0,1]$. □

Theorem 3.3. *For each $d\in\mathbb{N}$, let $\mathbf{M}^*_d$ be as in* (3.10) *and let $\mathbf{M}_d(\#\lambda)$ be the Hankel moment matrix associated with $\#\lambda$ (hence with sequence of moments as in* (3.3)*). Then :*

$$\phi^*_0 \;=\; \lim_{d\to\infty}\; \lambda_{\min}(\mathbf{M}_d(\#\lambda), \mathbf{M}^*_d), \tag{3.13}$$

i.e., ϕ_0^* is the limit of a sequence of minimum generalized eigenvalues associated with the pair $(\mathbf{M}_d(\#\lambda), \mathbf{M}_d^*)$, $d \in \mathbb{N}$.

Proof. By Proposition 3.2, $\mathbf{M}_d^* \succ 0$ for all $d \in \mathbb{N}$. For every $d \in \mathbb{N}$, let

$$\tau_d := \lambda_{\min}(\mathbf{M}_d(\#\lambda), \mathbf{M}_d^*) = \max\{\, \tau : \tau\, \mathbf{M}_d^* \preceq \mathbf{M}_d(\#\lambda)\,\} \text{ [by (2.1)]}, \tag{3.14}$$

which implies $\tau_d\, \mathbf{M}_d^* \preceq \mathbf{M}_d(\#\lambda)$. In addition, $\tau_d\, \mathbf{M}_{d,x(1-x)}^* \succeq 0$ follows from (3.12). On the other hand, as $\phi^* \le \#\lambda$, we also have $\phi_0^*\, \mathbf{M}_d^* = \mathbf{M}_d(\boldsymbol{\phi}^*) \preceq \mathbf{M}_d(\#\lambda)$ for all $d \in \mathbb{N}$. Hence, $\phi_0^* \le \tau_d$ for all $d \in \mathbb{N}$, and the sequence $(\tau_d)_{d\in\mathbb{N}}$ is monotone non increasing.

We next show that $\tau_d \downarrow \phi_0^*$ as $d \to \infty$. As $\tau_d \ge \phi_0^*$ for all d, $\lim_{d\to\infty} \tau_d = \tau^* \ge \phi_0^*$. Consider the sequence $\boldsymbol{\mu} = (\mu_j)_{j\in\mathbb{N}}$ defined by:

$$\mu_j = \tau^* \frac{n}{n+jt}, \quad j \in \mathbb{N}.$$

Then from $\tau_d\, \mathbf{M}_d^* \succeq 0$ for all d, and the convergence $\tau_d \to \tau^*$, we obtain

$$\tau^*\, \mathbf{M}_d^* = \mathbf{M}_d(\boldsymbol{\mu}) \succeq 0, \quad \forall d \in \mathbb{N}.$$

For similar reasons

$$\tau^*\, \mathbf{M}_{d,x(1-x)}^* = \mathbf{M}_d(x(1-x)\,\boldsymbol{\mu}) \succeq 0, \quad \forall d \in \mathbb{N}.$$

By Lemma 2.2, $\boldsymbol{\mu}$ is the moment sequence of a measure μ supported on $[0,1]$ with mass $\mu_0 = \mu([0,1]) = \tau^*$, and by construction we also have $\mu \le \#\lambda$. Therefore $\mu \in \mathscr{M}(S)$ is a feasible solution of (3.4) which implies $\mu([0,1]) \le \phi_0^*$. But on the other hand,

$$\phi_0^* \le \tau^* = \mu([0,1]) \le \phi_0^*,$$

which yields the desired result $\tau^* = \phi_0^*$. □

Algorithm 3.4 below provides the value τ_d of (3.14). Therefore to approximate $\mathrm{vol}(\mathbf{K})$ from above, apply **Algorithm 3.4** for a sequence $d = 1, 2, \ldots$. It produces the required monotone sequence of upper bounds $(\tau_d)_{d\in\mathbb{N}}$ on ϕ_0^*, which converges to $\phi_0^* = 2^{-n}\mathrm{vol}(\mathbf{K})$ as d increases.

Algorithm 3.4 (Computing τ_d in (3.14))**.**

(1) Compute all moments $\#\lambda_k$, $k = 1, \ldots, 2d$, of $\#\lambda$ by (3.3)
(2) Build up the Hankel matrices $\mathbf{M}_d^*$ and $\mathbf{M}_d(\#\lambda)$ in (3.10)
(3) Compute $\rho := \lambda_{\min}(\mathbf{M}_d(\#\lambda), \mathbf{M}_d^*)$ (e.g., by a standard linear algebra routine)
(4) Return ρ.

Finally, the following result shows that $\tau_d \le \rho_d$.

Proposition 3.5. *For each $d \in \mathbb{N}$, let ρ_d (resp. τ_d) be as in* (3.6) *(resp.* (3.14)*). Then $\rho_d \ge \tau_d$.*

Proof. Consider the sequence $\boldsymbol{\mu} = (\mu_j)_{j\le 2d}$ defined by:

$$\mu_j = \tau_d \frac{n}{n+jt}, \quad j \le 2d.$$

Then from (3.14), $\tau_d\, \mathbf{M}_d^* = \mathbf{M}_d(\boldsymbol{\mu})$ and therefore, $0 \preceq \mathbf{M}_d(\boldsymbol{\mu}) \preceq \mathbf{M}_d(\#\lambda)$. Similarly, $\tau_d\, \mathbf{M}_{d-1,x(1-x)} = \mathbf{M}_{d-1}(x(1-x)\,\boldsymbol{\mu}) \succeq 0$. In other words, the sequence $\boldsymbol{\mu}$ is a feasible solution of (3.6), which implies $\mu_0\, (= \tau_d) \le \rho_d$. □

Hence the above eigenvalue procedure (with no optimization involved) provides a monotone sequence of upper bounds on ϕ_0^*. Those upper bounds are better than the sequence of upper bounds $(\rho_d)_{d\in\mathbb{N}}$ obtained by solving the hierarchy of semidefinite relaxations (3.6). Notice also that the matrix $\mathbf{M}_d^*$ depends only on the degree "t" of g and not on g itself; see Proposition 3.2.

In fact there is a simple interpretation of this improvement. In Problem (3.4) and in its associated semidefinite relaxations (3.6), one may include the additional constraints

$$\phi_j \;=\; n\,\phi_0/(n+jt), \quad \forall j \le 2d, \tag{3.15}$$

coming from Stokes' theorem applied to ϕ^*; see Lemma 3.1. Indeed we are allowed to do that because ϕ^* (which is the unique optimal solution of (3.4)) satisfies these additional constraints. If it does not change the optimal value of (3.4), it changes that of (3.6) as it makes the corresponding relaxation stronger and therefore $\tau_d \le \rho_d$ for all d.

Remark 3.6. (i) If one uses the basis of orthonormal polynomials with respect to the pushforward measure $\#\lambda$ then the new moment matrix $\tilde{\mathbf{M}}_d(\#\lambda)$ (expressed in this basis) is the identity matrix. Therefore τ_d^{-1} is the maximum eigenvalue of the corresponding matrix $\tilde{\mathbf{M}}_d^*$ also expressed in that basis. This basis of orthonormal polynomials can be obtained from the decomposition $\mathbf{M}_d(\#\lambda) = \mathbf{D}\mathbf{D}^T$ for triangular matrices $\mathbf{D}$ and $\mathbf{D}^T$. For more details on multivariate orthogonal polynomials, the interested reader is referred to Dunkl and Xu [3] and the many references therein.

(ii) Alternatively one may also use an orthonormal basis associated with the sequence of moments $(n/(n+jt))_{j\in\mathbb{N}}$, i.e., a basis of polynomials that are orthonormal with respect to the measure ψ of Proposition 3.2. In this case the new moment matrix $\tilde{\mathbf{M}}_d^*$ expressed in this basis is the identity matrix and τ_d is now the minimum eigenvalue of the new moment matrix $\tilde{\mathbf{M}}_d(\#\lambda)$ expressed in this basis. Notice that the orthonormal basis depends on the degree t of g, and *not* on g itself.

(iii) Finally, another possibility is to simply use the basis of Chebyshev polynomials. Then computing τ_d is still solving a generalized eigenvalue problem but which involves matrices with a much better numerical conditioning.

Two straightforward extensions. A first extension is when g is *quasi-homogeneous*, i.e., when there exists $\mathbf{u} \in \mathbb{Q}^n$ such that $g(\lambda_1^{u_1}\mathbf{x}_1, \ldots, \lambda_1^{u_n}\mathbf{x}_n) = \lambda\, g(\mathbf{x})$ for all $\lambda > 0$ and all $\mathbf{x} \in \mathbb{R}^n$. Euler's identity becomes $\sum_i u_i \frac{\partial g(\mathbf{x})}{\partial x_i} x_i = g(\mathbf{x})$ for all $\mathbf{x}$. If one applies Stokes relation (2.3) with vector field $\mathbf{X} = (u_1x_1, \ldots, u_nx_n)$, then in Lemma 3.1 one obtains $\phi_j^* = \bar{u}\,\phi_0^*/(\bar{u}+j)$ with $\bar{u} = \sum_j u_j$. With $\mathbf{M}_d^*$ modified accordingly, Theorem 3.3 is still valid.

A second straightforward extension is to compute $\int_{\mathbf{K}} \mathbf{x}^\alpha\, d\lambda$ for a fixed $\alpha \in \mathbb{N}^n$, once sufficiently many moments (ϕ_j^*) of the measure ϕ^* in Lemma 3.1 have been approximated. As $\mathbf{K} \subset \mathbf{B} = [-1,1]^n$, consider the positive measure $d\lambda^\alpha := (1 - \mathbf{x}^\alpha)d\lambda$ on $\mathbf{B}$ and its associated pushforward measure $\#\lambda^\alpha$ by the mapping g. Its moments $(\#\lambda_j^\alpha)_{j\in\mathbb{N}}$ are also obtained in closed form easily. Let $\boldsymbol{\phi}^{\alpha *} = (\phi^{\alpha *})_{j\in\mathbb{N}}$ be the moments of the restriction of $\#\lambda^\alpha$ to $\mathbf{K}$. Using Stokes relation (2.3) with $\mathbf{X} = \mathbf{x}$ and $f = (1-g^j)(1-\mathbf{x}^\alpha)$, in the analogue of Lemma 3.1 one now obtains

the identity:

$$\phi_j^{\alpha *} \;=\; \frac{n+|\alpha|}{n+|\alpha|+tj}\,\phi_0^{\alpha *} + \frac{|\alpha|}{n+|\alpha|+tj}\,(\phi_j^* - \phi_0^*), \quad \forall\, j \in \mathbb{N}.$$

Again, every $\phi_j^{\alpha *}$ is expressed in terms of the *single* unknown $\phi_0^{\alpha *}$ since ϕ_0^* and ϕ_j^* are known. With $\mathbf{M}_d^*$ modified accordingly, an adapted version of Theorem 3.3 is still valid.

3.2. **Some numerical examples.** To show how this approximation of vol($\mathbf{K}$) from above by a sequence of eigenvalue problems of increasing size is much more efficient than solving the hierarchy of semidefinite programs (3.6) as suggested in [5], we have considered a favorable case for (3.6). We chose $\mathbf{K}$ to be the Euclidean unit ball $\{\mathbf{x} : \|\mathbf{x}\| \le 1\}$ with Lebesgue volume $\rho^* = \pi^{n/2}/\Gamma(1+n/2)$ and the box $\mathbf{B}$ that contains $\mathbf{K}$ is the smallest one, i.e., $\mathbf{B} = [-1,1]^n$. Indeed, the smaller is the box $\mathbf{B}$, the better are the approximation by ρ_d in (3.6).

We first describe the first two steps to appreciate the simplicity of the approach. Let $n = 2$ and $g = \|\mathbf{x}\|^2 = x_1^2 + x_2^2$, and $\mathbf{B} = [-1,1]^2$, so that vol($\mathbf{K}$) $= \pi$. Then:

$$\mathbf{M}_1^* \;=\; \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}; \quad \mathbf{M}_1(\#\lambda) \;=\; \begin{bmatrix} 1 & 2/3 \\ 2/3 & 28/45 \end{bmatrix}$$

This yields $4\cdot\tau_1 \approx 3.20$ which is already a good upper bound on π whereas $4\cdot\rho_1 = 4$.

$$\mathbf{M}_2^* \;=\; \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}; \quad \mathbf{M}_2(\#\lambda) \;=\; \begin{bmatrix} 1 & 2/3 & 28/45 \\ 2/3 & 28/45 & 24/35 \\ 28/45 & 24/35 & 2/9+8/21+6/25 \end{bmatrix}$$

This yields $4\cdot\tau_2 \approx 3.1444$ while $4\cdot\rho_2 = 3.8928$. Hence $4\tau_2$ already provides a very good upper bound on π with only moments of order 4. To appreciate the difference in speed of convergence between ρ_d and τ_d, Table 1 displays both values τ_d and ρ_d in the case of $n = 4$ variables and $d = 1,\ldots,5$. While the convergence $\tau_d \to 4.9348$ is quite fast with a relative error of 0.03% at step $d = 5$, the convergence $\rho_d \to 4.9348$ is extremely slow as $\rho_5 \approx 8.499$ only; see Figure 1

TABLE 1. $n = 4$, $\rho^* = 4.9348$; ρ_d versus τ_d

d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
$2^n\rho_d$	12.19	11.075	9.163	8.878	8.499
$2^n\tau_d$	6.839	5.309	5.001	4.945	4.936

We next provide results for the same problem but now in larger dimensions $n = 5, 8, 9, 10$ in Table 3, Table 4, Table 5, and Table 6 respectively. From inspection we can observe a fast and regular decrease in the value $2^n\tau_d$ as d increases, and similarly for the relative error.

TABLE 2. $n = 4$, $\rho^* = 4.9348$; τ_d and relative error

d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$2^n\tau_d$	6.839	5.309	5.001	4.945	4.936	4.935
$\frac{100(2^n\tau_d-\rho^*)}{\rho^*}$	38.6%	7.58%	1.35%	0.22%	0.03%	0.004%

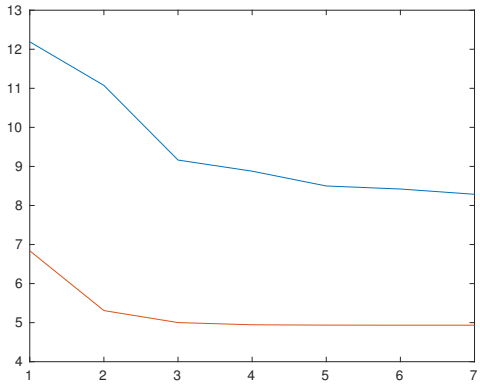


FIGURE 1. $n = 4$; Comparing τ_d (red below) with ρ_d (blue above)

TABLE 3. $n = 5$, $\rho^* = 5.26$; τ_d and relative error

d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$2^n\tau_d$	10.2892	6.5248	5.57	5.3347	5.2788	5.266
$\frac{100(2^n\tau_d-\rho^*)}{\rho^*}$	95%	23.95%	5.92%	1.34%	0.28%	0.05%

TABLE 4. $n = 8$, $\rho^* = 4.0587$; τ_d and relative error

d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$2^n\tau_d$	43.16	15.04	7.97	5.569	4.639	4.272	4.133	4.083
$\frac{100(2^n\tau_d-\rho^*)}{\rho^*}$	963%	270%	96%	37%	14%	5.26%	1.83%	0.60%

TABLE 5. $n = 9$, $\rho^* = 3.298$; τ_d and relative error

d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$2^n\tau_d$	73.406	21.682	9.801	5.935	4.413	3.764	3.485	3.369
$\frac{100(2^n\tau_d-\rho^*)}{\rho^*}$	2125%	557%	197%	79%	33.8%	14.1%	5.6%	2.15%

For $n = 10$ and $d = 8$, we have encountered numerical problems because the Hankel matrix $\mathbf{M}_8(\#\lambda)$ is ill-conditioned and then one should use another basis of polynomials in which to express the matrices $\mathbf{M}_8^*$ and $\mathbf{M}_8(\#\lambda)$; see Remark 3.6.

Influence of the size of the box B. If one increases the size of the box $\mathbf{B} = [-r, r]^n$ that contains $\mathbf{K}$ then one expects a slower convergence and this is why it is recommended to take for $\mathbf{B}$ the smallest box that contains $\mathbf{K}$. An appropriate choice is the box $\prod_{i=1}^n[-u_i, u_i]$ where u_i (resp. v_i) is a lower bound (resp. upper bound) as close as possible to $\min\{\, x_i : \mathbf{x} \in \mathbf{K}\,\}$ (resp. $\max\{x_i : \mathbf{x} \in \mathbf{K}\,\}$), which can be computed by the first step of the Moment-SOS hierarchy described in [9]. From results displayed in Table 7 with $r = 1$ and $r = 1.3$, one observes that even though the convergence is a bit slower, it is still quite good. The initial value τ_1 is significantly higher but then τ_d (with $r = 1.3$) still decreases very fast; see Figure 2.

TABLE 6. $n = 10$, $\rho^* = 2.55$; τ_d and relative error

d	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$2^n\tau_d$	32.432	12.657	6.662	4.375	3.379	2.921
$\frac{100(2^n\tau_d-\rho^*)}{\rho^*}$	1171%	396.3%	161%	71.6%	32.5%	14.54%

TABLE 7. $n = 5$; $\mathrm{vol}(\mathbf{K}) = 5.2638$; Influence of the size of $\mathbf{B} = [-r, r]^n$ with $r = 1$ and $r = 1.3$

d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$r=1.3$; $(2r)^n\tau_d$	26.345	11.744	7.622	6.149	5.585	5.373	5.299	5.275
$r=1$; $(2r)^n\tau_d$	10.289	6.524	5.575	5.334	5.278	5.266	5.264	5.2639

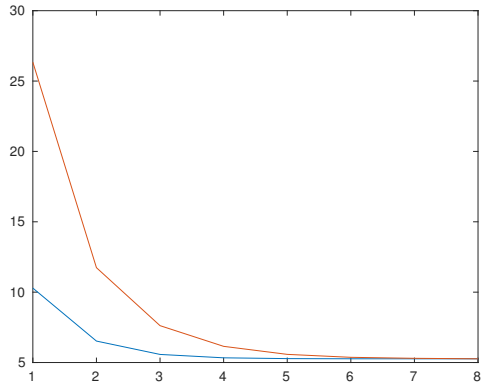


FIGURE 2. $n = 8$; Comparing τ_d with $r = 1.3$ (red above) and $r = 1$ (blue below)

4. EXTENSIONS

In this section we discuss two extensions of the above methodology, when:

- $\mathbf{K} := \{\mathbf{x} : a \le g(\mathbf{x}) \le b\} \subset (-1, 1)^n$ and g is not homogeneous anymore.
- $\mathbf{K}$ is now $\{\mathbf{x} : g_j(\mathbf{x}) \ge 0,\ j = 1, \ldots, m\} \subset (-1, 1)^n$ and each g_j is homogeneous (with one of them being nonnegative).

In the second extension, again following Jasour et al. [5], one considers the push-forward of the Lebesgue measure by the polynomial mapping $g : \mathbf{B} \to \mathbb{R}^m$ which maps $\mathbf{x} \in \mathbf{B}$ to the vector $(g_j(\mathbf{x}))_{j=1}^m \in \mathbb{R}^m$. Then the initial Lebesgue volume computation in $\mathbb{R}^n$ is reduced to an equivalent "measure" computation problem of the form (3.8) but now in $\mathbb{R}^m$ (instead of $\mathbb{R}$). One may apply the hierarchy of semidefinite programs described in [4]. But as we did in §3, we can exploit again the homogeneity of the g_j's to strengthen the semidefinite relaxations defined in [5], by introducing additional linear constraints coming from an appropriate application of Stokes' theorem. The only difference with the univariate case treated in §3 is that the problem is not an eigenvalue problem any more.

The first extension to the non homogeneous case is perhaps more interesting. We now write g as a sum of homogeneous polynomials of increasing degree 1,

$2,\ldots,\deg(g)$, and consider again a pushforward of the Lebesgue mesure λ by the polynomial mapping $g:\mathbf{B}\to\mathbb{R}^{\deg(g)}$, $\mathbf{x}\mapsto(g_1(\mathbf{x}),\ldots,g_{\deg(g)}(\mathbf{x}))$.

4.1. **The non-homogeneous case.** Let $\mathbf{B}=[-1,1]^n$, and suppose that $\mathbf{K}\subset\mathbb{R}^n$ is now described by:

$$\mathbf{K}\ :=\ \{\,\mathbf{x}:a\ \le\ g(\mathbf{x})\ \le\ b\,\},\tag{4.1}$$

for some $a,b\in\mathbb{R}$, where $g\in\mathbb{R}[\mathbf{x}]_t$, and $\mathbf{K}\subset(-1,1)^n$, possibly after scaling. With no loss of generality we may and will assume that $g(0)=0$ and write

$$\mathbf{x}\mapsto g(\mathbf{x})\ =\ \sum_{k=1}^{t} g_k(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^n,$$

where for each $1\le k\le t$, g_k is a homogeneous polynomial of degree k.

We next see how to adapt the previous methodology of §3 to this more general case in a relatively simple manner. To simplify the exposition and alleviate notation, we describe the quadratic case $t=2$. It will become obvious to understand how to proceed for $t>2$. So with $t=2$, $g=g_1+g_2$ with g_1 (resp. g_2) homogeneous of degree 1 (resp. 2).

Consider the pushforward measure $\#\lambda$ on $\mathbb{R}^2$ of λ on $\mathbf{B}$, by the polynomial mapping:

$$g:\mathbf{B}\to\mathbb{R}^2,\quad \mathbf{x}\mapsto g(\mathbf{x})=\begin{bmatrix}g_1(\mathbf{x})\\ g_2(\mathbf{x})\end{bmatrix},\quad \mathbf{x}\in\mathbf{B}.$$

Let $\Theta:=g(\mathbf{B})\subset\mathbb{R}^2$ be the support of the pushforward measure $\#\lambda$, and observe that for each $i,j\in\mathbb{N}$:

$$\#\lambda_{ij}\ :=\ \int_\Theta z_1^i z_2^j\,d\#\lambda(\mathbf{z})\ =\ \int_{\mathbf{B}} g_1(\mathbf{x})^i\,g_2(\mathbf{x})^j\,d\lambda(\mathbf{x}),\tag{4.2}$$

can be obtained in closed form. Letting

$$S\ :=\ g(\mathbf{K})\ =\ \{\mathbf{z}\in\Theta:\ a\ \le\ z_1+z_2\ \le b\,\},$$

we obtain $2^{-n}\mathrm{vol}(\mathbf{K})=\#\lambda(S)$. Next, recall that (see (3.4) in §3):

$$2^{-n}\mathrm{vol}(\mathbf{K})\ =\ \#\lambda(S)\ =\ \max_{\phi\in\mathscr{M}(S)}\{\,\phi(S):\ \phi\ \le\ \#\lambda\,\}\tag{4.3}$$

and ϕ^* is the unique optimal solution of (4.3).

Let $\mathbf{z}\mapsto\tilde{h}(\mathbf{z}):=(b-z_1-z_2)(z_1+z_2-a)$. The hierarchy of semidefinite relaxations associated with (4.3) and indexed by $d\in\mathbb{N}$, read:

$$\rho_d\ =\ \max_{\boldsymbol{\phi}}\{\,\phi_0:\ 0\ \preceq\mathbf{M}_d(\boldsymbol{\phi})\ \preceq\ \mathbf{M}_d(\#\lambda);\quad \mathbf{M}_{d-1}(\tilde{h}\,\boldsymbol{\phi})\ \succeq 0\,\},\tag{4.4}$$

where the maximization is over finite bivariate sequences $\boldsymbol{\phi}=(\phi_{ij})_{i+j\le 2d}$. Again invoking [4], $\rho_d\downarrow 2^{-n}\mathrm{vol}(\mathbf{K})$ as $d\to\infty$.

Next, following the same philosophy as in §3, we are going to use some additional information on the optimal solution ϕ^* of (4.3) to strengthen the semidefinite relaxations (4.4). To do so we again use Stokes' theorem.

Stokes. Recall that $\mathbf{K} \subset (-1,1)^n$ and therefore, $\partial\mathbf{K} \subset \{\mathbf{x} \in \mathbf{B} : h(\mathbf{x}) = 0\}$ where $\mathbf{x} \mapsto h(\mathbf{x}) := (b - g_1(\mathbf{x}) - g_2(\mathbf{x}))(g_2(\mathbf{x}) + g_2(\mathbf{x}) - a)$. Therefore by Stokes' theorem,

$$\begin{aligned} 0 &= n \int_{\mathbf{K}} g_1(\mathbf{x})^i \, g_2(\mathbf{x})^j \, h(\mathbf{x}) \, d\lambda(\mathbf{x}) \\ &\quad + \int_{\mathbf{K}} \langle \mathbf{x}, \nabla(g_1(\mathbf{x})^i \, g_2(\mathbf{x})^j \, h(\mathbf{x})) \, d\lambda(\mathbf{x}), \quad \forall i, j \in \mathbb{N}. \end{aligned} \tag{4.5}$$

Developing and using homogeneity of g_1, g_2, one obtains:

$$\begin{aligned} 0 &= n \int_S \Big[(n + i + 2j) \, z_1^i z_2^j \, (b - z_1 - z_2)(z_1 + z_2 - a) \\ &\quad + (z_1 + 2z_2) \, z_1^i z_2^j \, (a + b - 2z_1 - 2z_2) \Big] . \, d\#\lambda(\mathbf{z}). \end{aligned}$$

Equivalently, for every $i, j \in \mathbb{N}$, introduce the polynomial $q_{ij} \in \mathbb{R}[\mathbf{z}]$:

$$\begin{aligned} \mathbf{z} \mapsto q_{ij}(\mathbf{z}) &:= (n + i + 2j) \, z_1^i z_2^j \, (b - z_1 - z_2)(z_1 + z_2 - a) \\ &\quad + (z_1 + 2z_2) \, z_1^i z_2^j \, (a + b - 2z_1 - 2z_2). \end{aligned} \tag{4.6}$$

Then one obtains:

$$\int_S q_{ij}(\mathbf{z}) \, d\phi^*(\mathbf{z}) \, = \, 0, \quad \forall \, i, j \in \mathbb{N}. \tag{4.7}$$

Notice that (4.7) is a linear relation between moments of ϕ^*, the optimal solution of (4.3). That is, let $\boldsymbol{\phi}^* = (\phi^*_{ij})_{i,j \in \mathbb{N}}$ be the sequence of moments of ϕ^* on S, and let $L_{\boldsymbol{\phi}^*} : \mathbb{R}[\mathbf{z}] \to \mathbb{R}$ be the Riesz functional

$$q \, (= \sum_{i,j} q_{ij} z_1^i \, z_2^j) \quad \mapsto L_{\boldsymbol{\phi}^*}(q) \, := \sum_{i,j} q_{ij} \, \phi^*_{ij}, \quad q \in \mathbb{R}[\mathbf{z}].$$

Then (4.7) reads

$$L_{\boldsymbol{\phi}^*}(q_{ij}) \, = \, 0, \quad i, j \in \mathbb{N}, \tag{4.8}$$

So we can strengthen the relaxations (4.4) by adding the additional "Stokes" moments constraints (4.8), that is, for every d one solves the semidefinite program:

$$\begin{aligned} \tau_d \, = \, \max_{\boldsymbol{\phi}} \{ \, \phi_0 : \quad & 0 \preceq \mathbf{M}_d(\boldsymbol{\phi}) \preceq \mathbf{M}_d(\#\lambda); \quad \mathbf{M}_{d-1}(\tilde{h} \, \boldsymbol{\phi}) \succeq 0; \\ & L_{\boldsymbol{\phi}^*}(q_{ij}) \, = \, 0, \quad \text{for all } (i,j) \text{ s.t. } \deg(q_{ij}) \le 2d \, \}, \end{aligned} \tag{4.9}$$

which is clearly a strengthening of (4.4).

Proposition 4.1. *Let ρ_d (resp. τ_d) be as in* (4.4) *(resp.* (4.9)*), $d \in \mathbb{N}$. Then:*

$$\#\lambda(S) \, \le \, \tau_d \, \le \rho_d \quad \textit{for all } d, \textit{ and } \tau_d \downarrow \#\lambda(S) \textit{ as } d \textit{ increases.} \tag{4.10}$$

Proof. That $\tau_d \le \rho_d$ for all d, is straightforward and similarly for the monotonicity of the sequence $(\tau_d)_{d \in \mathbb{N}}$. Next, as ϕ^* is the optimal solution of (4.3), its sequence of moments $\boldsymbol{\phi}^* = (\phi^*_{ij})$ is feasible for (4.9), with associated value $\phi^*_0 = \#\lambda(S) = 2^{-n}\mathrm{vol}(\mathbf{K})$. Hence $\tau_d \ge \#\lambda(S)$. Then the convergence $\tau_d \downarrow \#\lambda(S)$ follows from $\rho_d \downarrow \#\lambda(S)$. □

The difference with the homogeneous case treated in §3 is that now computing τ_d requires solving the semidefinite program (4.9) whereas in §3 computing τ_d reduces to solving a generalized eigenvalue problem, hence with no optimization involved. However notice that instead of solving the costly n-variate semidefinite relaxations associated with (3.8) in $\mathbb{R}^n$, we now solve similar semidefinite relaxations but for

a *bivariate* problem on the plane. In addition the (convergence) acceleration technique based on Stokes's theorem can also be implemented; see (4.9).

4.2. **Multi-homogeneous constraints.** Another extension is when $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \le 1,\ j = 1,\ldots,m\} \subset (-1,1)^n$ for a family $(g_j)_{j=1}^m$ of homogeneous polynomials, not necessarily of same degree, say $\deg(g_j) = t_j$, and at least one of them is positive on $(\mathbb{R}\setminus\{0\})^n$. In this case one may proceed again as suggested in Jasour et al. [5]. Now $\#\lambda$ is the pushforward on $\mathbb{R}^m$ of λ on $\mathbf{B}$, by the mapping:

$$g : \mathbf{B} \to \mathbb{R}^m, \quad g(\mathbf{x}) \;=\; \begin{bmatrix} g_1(\mathbf{x}) \\ \cdots \\ g_m(\mathbf{x}) \end{bmatrix}.$$

In particular the moments of $\#\lambda$ are defined by:

$$\#\lambda_\alpha \;=\; \int_{\mathbf{B}} g_1(\mathbf{x})^{\alpha_1}\cdots g_m(\mathbf{x})^{\alpha_m}\,\lambda(d\mathbf{x}) \;=\; \int_{g(\mathbf{B})} \mathbf{z}^\alpha\,\#\lambda(d\mathbf{z}), \quad \forall \alpha \in \mathbb{N}^m.$$

Again all moments $\#\lambda_\alpha$ can be computed in closed form, and with $S = [0,1]^m$, $2^{-n}\mathrm{vol}(\mathbf{K}) = \#\lambda(S)$. Let us describe how the generalization works for the case $m = 2$. Again denote by ϕ^* on $\mathbb{R}^2$ the restriction of $\#\lambda$ to S and let $\boldsymbol{\phi}^* = (\phi^*_{ij})_{i,j\in\mathbb{N}}$ with:

$$\phi^*_{ij} \;:=\; \int_S z_1^i z_2^j\,\phi^*(d\mathbf{z}), \quad \forall i,j = 0,1,\ldots.$$

So the bivariate analogues of the semidefinite relaxations (3.6) read:

$$\begin{aligned} \rho_d = \max_{\boldsymbol{\phi}}\{\,\phi_0 :\ & 0 \preceq \mathbf{M}_d(\boldsymbol{\phi}) \preceq \mathbf{M}_d(\#\lambda) \\ & \mathbf{M}_{d-1}(x_j(1-x_j)\,\boldsymbol{\phi}) \succeq 0, \quad j = 1,2\,\}, \end{aligned} \tag{4.11}$$

where $\boldsymbol{\phi} = (\phi_{ij})_{i+j\le 2d}$, and $\mathbf{M}_d(\boldsymbol{\phi})$ (resp. $\mathbf{M}_{d-1}(x_j(1-x_j)\,\boldsymbol{\phi})$, $j = 1,2$) is the moment (resp. localizing) matrix associated with $\boldsymbol{\phi}$ (resp. with $\boldsymbol{\phi}$ and $\mathbf{x}\mapsto x_j(1-x_j)$, $j = 1,2$). Then $\rho_d \downarrow \#\lambda(S)$ as $d\to\infty$. Again the semidefinite relaxations (4.11) are a lot cheaper to solve than those associated with the n-variate problem (3.8).

As we did for the univariate case we can improve the above convergence by adding additional constraints that must be satisfied at the optimal solution ϕ^* of (3.8). Again $\phi^*_0 = 2^{-n}\mathrm{vol}(\mathbf{K})$. Let $(i,j,k,\ell)\in\mathbb{N}^4$ with $k,\ell\ge 1$. Then with $X(\mathbf{x}) = \mathbf{x}$, Stokes's Theorem yields

$$\begin{aligned}
0 &= n\int_{\mathbf{K}} g_1^i g_2^j\,(1-g_1)^k(1-g_2)^\ell\lambda(d\mathbf{x})\\
&\quad+\int_{\mathbf{K}}\langle\mathbf{x},\nabla[g_1^i g_2^j(1-g_1)^k(1-g_2)^\ell]\rangle\,\lambda(d\mathbf{x})\\
&= n\int_S z_1^i z_2^j\,(1-z_1)^k(1-z_2)^\ell\,\#\lambda(d\mathbf{z})\\
&\quad+it_1\int_S z_1^i z_2^j\,(1-z_1)^k(1-z_2)^\ell\,\#\lambda(d\mathbf{z})\\
&\quad+jt_2\int_S z_1^i z_2^j\,(1-z_1)^k(1-z_2)^\ell\,\#\lambda(d\mathbf{z})\\
&\quad-kt_1\int_S z_1^{i+1} z_2^j\,(1-z_1)^{k-1}(1-z_2)^\ell\,\#\lambda(d\mathbf{z})\\
&\quad-\ell t_2\int_S z_1^i z_2^{j+1}(1-z_1)^k(1-z_2)^{\ell-1}\,\#\lambda(d\mathbf{z}).
\end{aligned}$$

That is, for each $(i,j,k,\ell)\in\mathbb{N}^4$ with $k,\ell\geq 1$, one obtains a linear constraint that links some moments of ϕ^*, that we denote by $L_{\boldsymbol{\phi}^*}(q_{ijk\ell})$ where $q_{ijk\ell}\in\mathbb{R}[\mathbf{z}]$ is the above polynomial under the integral sign. For instance,

$$0=L_{\boldsymbol{\phi}^*}(q_{0011})=n(\phi^*_0-\phi^*_{10}-\phi^*_{01}+\phi^*_{11})-t_1(\phi^*_0-\phi^*_{01})-t_2(\phi^*_0-\phi^*_{10}).$$

$0=L_{\boldsymbol{\phi}^*}(q_{1111})=(n+t_1+t_2)\,(\phi^*_{11}+\phi^*_{22}-\phi^*_{21}-\phi^*_{12})-t_1(\phi^*_{21}-\phi^*_{22})-t_2(\phi^*_{12}-\phi^*_{22})$, etc. So we can add these additional constraints to (4.11) and solve:

$$\begin{aligned}
\tau_d=\max_{\boldsymbol{\phi}}\{\,\phi_0:\quad & 0\preceq\mathbf{M}_d(\boldsymbol{\phi})\preceq\mathbf{M}_d(\#\lambda)\\
& \mathbf{M}_{d-1}(x_i(1-x_j)\,\boldsymbol{\phi})\succeq 0,\quad j=1,2\\
& L_{\boldsymbol{\phi}}(q_{i,j,k,\ell})=0,\quad k,\ell\geq 1;\ i+j+k+\ell\leq 2d\,\}.
\end{aligned}\tag{4.12}$$

Of course $\tau_d\leq\rho_d$ for all d and therefore $\tau_d\downarrow\#\lambda(S)$ as d increases. Again, the difference with the univariate case is that now computing τ_d requires to solve the semidefinite program (4.12) instead of a generalized eigenvalue problem. However it is of same dimension as (4.11) and the convergence $\tau_d\downarrow\#\lambda(S)$ is expected to be much faster than $\rho_d\downarrow\#\lambda(S)$ as we have been able to include additional constraints based on Stokes' theorem.

5. Conclusion

We have presented a new methodology to approximate (in principle as closely as desired) the Lebesgue volume of the sublevel set $\{\mathbf{x}:g(\mathbf{x})\leq 1\}$ of a positive multivariate polynomial g. Inspired by Jasour et al. [5], we formulate an equivalent "volume" computation $\mu(I)$ of an interval I of the real line for a certain pushforward measure μ. The novelty with respect to [5] is that by using Stokes' theorem and exploiting the homogeneity of g, we are able to further reduce the problem to solving a hierarchy of generalized eigenvalue problems for Hankel matrices of increasing size, with *no* optimization involved. To the best of our knowledge, this characterization of Lebesgue volume as the limit of eigenvalue problems of increasing size is new. Moreover the methodology also extends to sublevel sets of arbitrary polynomials. It then reduces the Lebesgue volume computation in $\mathbb{R}^n$ to a "volume" computation in $\mathbb{R}^d$ for a certain pushforward measure, where d is the degree of

the initial polynomial. An extension to several homogeneous constraints has been also described with a pattern similar to the extension for a single non-homogeneous constraint. Preliminary results on a simple case reveal a drastic improvement on the approximation scheme proposed in [5].

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