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Matthieu Barreau[a,*], Sophie Tarbouriech[b], Frédéric Gouaisbaut[b]

[a]*Division of Decision and Control Systems, KTH Royal Institute of Technology Stockholm, Sweden.*
[b]*LAAS-CNRS, Université de Toulouse, CNRS, UPS, Toulouse, France.*

Abstract

This paper deals with the stability analysis of a mass-spring system subject to friction using Lyapunov-based arguments. As the described system presents a stick-slip phenomenon, the mass may then periodically stick to the ground. The objective consists in developing numerically tractable conditions ensuring the global asymptotic stability of the unique equilibrium point. The approach proposed merges two intermediate results: The first one relies on the characterization of an attractor around the origin, in which converge the closed-loop trajectories. The second result assesses the regional asymptotic stability of the equilibrium point by estimating its basin of attraction. The main result relies on conditions allowing to ensure that the attractor issued from the first result is included in the basin of attraction of the origin computed form the second result. An illustrative example draws the interest of the approach.

Keywords: Friction, Lyapunov methods, Attractor, Regional asymptotic stability, Global asymptotic stability, LMI.
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*Corresponding author
Email addresses: `barreau@kth.se` (Matthieu Barreau), `tarbouriech@laas.fr` (Sophie Tarbouriech), `fgouaisb@laas.fr` (Frédéric Gouaisbaut)

1. Introduction

Friction appears in many mechanical systems such as drill-strings [1, 2], car steering [3] or also machine positioning [4, 5]. This is a nonlinear force which is responsible of many undesirable effects such as stick-slip or hunting [4]. The first challenge was then to propose a model which was generic enough to be easily adapted to a new situation and able to reproduce these phenomena.

Many models were investigated in the last century of different complexities with a relatively good correlation with empirical data. The first model was proposed by Guillaume Amontons and Charles Augustin de Coulomb [6] during the eighteenth century. Then, it was studied more precisely by Stribeck [7] who experimentally observed a decrease of the friction force at low velocity. Then many models arose trying to fit with the experiments conducted by Stribeck such as the Dahl model [8], the LuGre model [9] or Leuven friction model [10]. The survey [4] and the paper [11] draw comparisons on several models from simulation and experimental points of view.

Before designing controllers (see [5, 12, 13]) which were able to reduce the undesirable effects, it was needed to provide analysis tools to quantify the amplitudes of the induced oscillations [14]. This challenge gave rise to many techniques quantifying the nonlinear behavior introduced by the nonlinear term. For an empirical analysis, one can refer to [15]. An approximation based method like the describing method is studied in [2, 16]. An investigation of the analytical solution is conducted in [17] and Lyapunov methods are used in [18]. Few of the previous cited works are providing an exact stability test which is reliable with a low computational burden, and, to the best of our knowledge, there does not exist any regional (local) stability analysis.

This paper deals with the stability analysis of a mass-spring system subject to friction by taking into account that the described system presents a stick-slip phenomenon, meaning that the mass is periodically stick to the ground. This paper focuses on deriving numerically tractable conditions ensuring the global asymptotic stability of the origin. The approach proposed to attain this objective follows an alternative route to that one developed in [12], [19]. Indeed, the idea is to combine two intermediate results, the first one relying on a global convergence property to an attractor around the equilibrium point, whereas the second one focuses on the estimation of the basin of attraction of the equilibrium point. Hence, the first result concerns the characterization of an attractor around the equilibrium point, in which converge the closed-loop trajectories, using Lyapunov-based arguments. The second result studies the regional asymptotic stability of the equilibrium point and proposes an estimation of the basin of attraction of the equilibrium point. The main result expands the two sets of previous conditions in order to ensure that the attractor issued from the first result is included in the basin of attraction of the origin computed from the second result. Moreover, we provide conditions related to the system physics to characterize the case when the conditions for global asymptotic stability of the origin are feasible. An illustrative example shows the key strengths and the drawbacks of the proposed technique.

The paper is organized as follows. Section 2 is devoted to the description of the physical setup and the problem statement. In Section 3, the characterization of the attractor around the origin is presented. In Section 4, an inner-estimate of the basin of attraction of the origin is derived. The main result dealing with the global asymptotic stability of the origin is given in Section 5. Section 6 illustrates the effectiveness of the proposed approach. Finally, Section 7 ends the paper emphasizing possible perspectives.

Notation. $\mathbb{R}^{n\times m}$ stands for the set of all $n \times m$ real matrices. $P \in \mathbb{S}^n_+$ or equivalently $P \succ 0$ denotes a symmetric positive-definite matrix of $\mathbb{R}^{n\times n}$. For any matrices A and B, we define the operation $\mathrm{He}(A) = A + A^\top$. The notation I_n denotes the n by n identity matrix. We define the operation $\mathrm{col}(u, v) = \begin{bmatrix} u^\top & v^\top \end{bmatrix}^\top$ for any column vectors u and v. The Euclidean norm of a column-vector $u \in \mathbb{R}^n$ is $\|u\| = \sqrt{u^\top u}$.

2. Model description and problem formulation

2.1. Physical setup

The setup consists in studying the position and the velocity of a mobile of mass m, connected to a spring with an end moving forward at a constant speed $\dot{x}_A = v_{ref}$, parallel to the direction of movement. This is described in Figure 1.

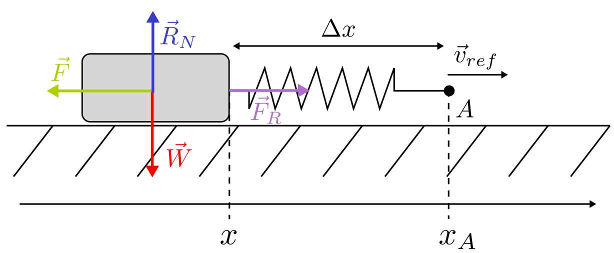


Figure 1: Physical setup: mass with a spring. The forces applied on the mass are in color and the point A has the speed v_{ref}.

The forces applied to the mass are the following ones:

1. the weight $\vec{W} = m\vec{g}$ where $\vec{g}$ is the gravitational acceleration;
2. the normal force $\vec{R}_N = -m\vec{g}$;
3. the elastic force $\vec{F}_R = k\,(x_A - x)$;
4. and the friction force $\vec{F}$.

In this work, the simplest friction model is considered. It was introduced firstly by Karnopp [20] and has shown a good correlation with the experimental data. It was then investigated more deeply in [15], and its final mathematical formulation is as follows:

$$\begin{aligned} F(\dot{x}(t)) &= R_N \left(\mu_C + (\mu_S - \mu_C) e^{-\left|\frac{\dot{x}(t)}{v_s}\right|^2} \right) \mathrm{Sign}(\dot{x}(t)) + k_v \dot{x}(t), \\ &= F_{nl}(\dot{x}(t)) + k_v \dot{x}(t), \end{aligned} \tag{1}$$

where $R_N = mg$, v_s is a positive constant, μ_S and μ_C positive constants such that $\mu_S - \mu_C > 0$, and the Sign function is defined as:

$$\mathrm{Sign}(\theta) = \begin{cases} \mathrm{sign}(\theta) = \frac{\theta}{|\theta|} & \text{if } \theta \neq 0, \\ [-1, 1] & \text{if } \theta = 0, \end{cases} \tag{2}$$

for $\theta \in \mathbb{R}$. Note that the Sign is defined as a set-valued function as in [5, 12] and can be seen as the convex envelop of the classical sign function.

The previous expression for the friction force can be split into three important contributions:

1. the Coulomb force: $\mu_C \left(1 - e^{-\frac{\dot{x}^2(t)}{v_s^2}}\right) R_N \mathrm{Sign}(\dot{x}(t))$;
2. the static friction force: $\mu_S e^{-\frac{\dot{x}^2(t)}{v_s^2}} R_N \mathrm{Sign}(\dot{x}(t))$;
3. and the viscous friction: $k_v \dot{x}(t)$.

The Coulomb force is the friction force acting at relatively high speed while the static friction occurs for low velocity. The viscous friction is a linear term which is related mostly to the friction with the air. A chart of this function is proposed in Figure 2 to observe the Stribeck curve, that is the non-monotonicity of the function F for positive speed.

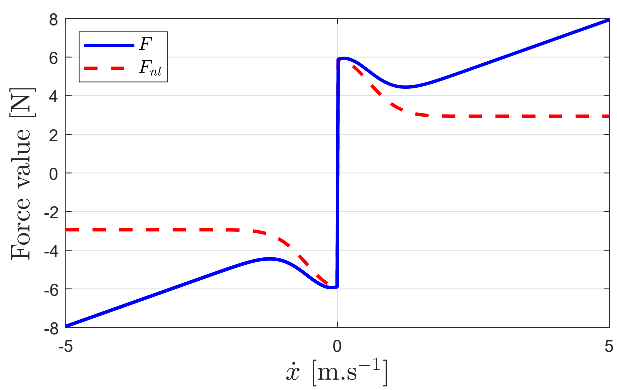


Figure 2: Chart of the friction force (in solid blue) and static and Coulomb forces (in dashed red) in our problem. Note that the friction force is not a monotonous function of the speed, inducing the famous stick-slip effect.

We study here the evolution of $\dot{x}$ and more particularly, its asymptotic behavior. Assuming $x_A(0) = 0$ and applying the second law of motion leads to the following dynamical equation for $t \geq 0$:

$$\begin{cases} \ddot{x}(t) = -\frac{1}{m} F(\dot{x}(t)) - \frac{k}{m}\left(x(t) - v_{ref} t\right), \\ x(0) = 0, \quad \dot{x}(0) = v_0, \end{cases} \tag{3}$$

where $v_0 \in \mathbb{R}$. To ease the reading, the previous system is not written using a differential inclusion, which is induced by the set-valued mapping sign function as defined in (2).

2.2. Problem Statement

We introduce the following state variables: $v = \dot{x}$ and $z(t) = x(t) - v_{ref}t$ such as (3) reads for $t > 0$ as:

$$\begin{bmatrix} \dot{v}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{m}\left(F_{nl}(v(t)) + k_v v(t) + kz(t)\right) \\ v(t) - v_{ref} \end{bmatrix}, \tag{4}$$

with $v(0) = v_0$, $z(0) = z_0$.

Remark 1. *Using a similar reasoning than in [12, 21, 19], one can conclude that there exists a unique solution to system* (4).

We are now looking for the equilibrium points (v_∞, z_∞) of (4). Assuming $k \cdot v_{ref} \neq 0$, there is a unique equilibrium point and we easily get:

$$\begin{cases} v_\infty = v_{ref}, \\ z_\infty = -\dfrac{F(v_{ref})}{k} = -\dfrac{F_{nl}(v_{ref}) + k_v v_{ref}}{k}. \end{cases}$$

Defining the error variable as follows:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} v - v_{ref} \\ z - z_\infty \end{bmatrix},$$

and $\phi_{v_{ref}}(\varepsilon_1) = F_{nl}(\varepsilon_1 + v_{ref}) - F_{nl}(v_{ref})$, the error dynamic is described by the dynamical system:

$$\dot{\varepsilon}(t) = A\varepsilon(t) + B\phi_{v_{ref}}(\varepsilon_1(t)) \tag{5}$$

with $\varepsilon_1(0) = v_0 - v_{ref}$, $\varepsilon_2(0) = z_0 - z_\infty$,

$$A = \begin{bmatrix} -\frac{k_v}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{1}{m} \\ 0 \end{bmatrix}.$$

From practical experiments, one can observe that due to the presence of the nonlinearity $\phi_{v_{ref}}(\varepsilon_1)$, it can be impossible to guarantee the global asymptotic convergence of the trajectories to the origin and some limit cycle around the origin can exist [15, 22]. Then, for the class of systems as described by (5), it is of interest to characterize a region $\mathcal{A}_{v_{ref}}$ of the state space, containing the origin and the possible limit cycle, which is assured to be a global attractor of system (5). Then the first problem we intend to address is the following.

Problem 1. *Characterize an outer-estimation of the global attractor $\mathcal{A}_{v_{ref}}$ of system* (5)*.*

For some cases, one can observe that the trajectories of system (5) converge to the origin. Then, it appears possible to estimate the region of initial conditions for which the asymptotic stability of the origin will be obtained. In this sense, the second problem we are interested in, relies on regional asymptotic stability of the origin. Note that since $\phi_{v_{ref}}$ is continuous around $\phi_{v_{ref}}(0) = 0$, there might exist a basin of attraction around the origin. But in order to be able to estimate it we need to study what happens for the linearized version of system (5), and therefore what happens with the derivative of ϕ with respect to ε_1 denoted $\frac{\partial \phi}{\partial \varepsilon_1}$ [23]. Then the second problem we intend to address is the following.

Problem 2. *Characterize an inner-estimation of the basin of attraction of the origin, denoted $\mathcal{D}_{v_{ref}}$ for system* (5)*.*

Unlike Problem 1, to address this problem, we need to find conditions depending on v_{ref}.

Finally, the main problem we want to solve is to provide conditions in order to ensure that the origin will be a globally asymptotically stable equilibrium point. To do this, the approach will consist in combining the solutions to Problems 1 and 2. Hence, if we are able to provide conditions such that $\mathcal{A}_{v_{ref}} \subseteq \mathcal{D}_{v_{ref}}$, then there does not exist a limit cycle and the origin will be globally asymptotically stable. Thus the main problem we intend to address is the following.

Problem 3. *Derive conditions for the outer-estimation of the attractor to be* *included in the inner-estimation of the basin of attraction of the origin, that is,* $\mathcal{A}_{v_{ref}} \subseteq \mathcal{D}_{v_{ref}}$.

3. Global Attractor Characterization

In order to address Problem 1, one first proposes a way to encapsulate the nonlinearity $\phi_{v_{ref}}$. Since F_{nl} is an odd function and F_{nl}^2 is lower and upper bounded almost everywhere by $F_C^2 = \mu_C^2 m^2 g^2$ and $F_S^2 = \mu_S^2 m^2 g^2$ respectively as shown in Figure 3, the function F_{nl} can be encapsulated into two relays.

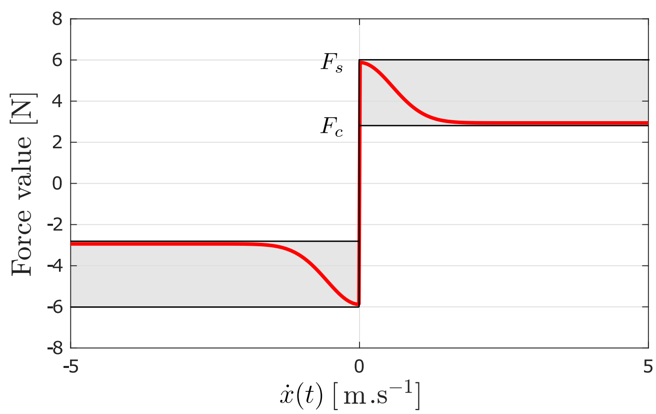


Figure 3: Encapsulation of F_{nl}.

Consequently, the following lemma can be proposed.

Lemma 1. *For any* $\varepsilon_1 \in \mathbb{R}$ *the following inequalities hold:*

$$\begin{aligned} F_{nl}(\varepsilon_1 + v_{ref})^2 &\leq F_S^2, \\ -2(\varepsilon_1 + v_{ref})F_{nl}(\varepsilon_1 + v_{ref}) &\leq 0. \end{aligned}$$

In the case $\varepsilon_1 \neq -v_{ref}$, *we also have:*

$$F_C^2 \leq F_{nl}(\varepsilon_1 + v_{ref})^2.$$

By using Lemma 1, the following theorem proposes a solution to Problem 1.

Theorem 1. *If there exist* $P_g \in \mathbb{S}_+^2$ *and* $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \geq 0$ *such that the following matrix inequalities hold:*

$$\mathrm{He}\left(D^\top P_g F\right) - \tau_0 \Pi_0 - \tau_1 \Pi_1 - \tau_2 \Pi_2 - \tau_3 \Pi_3 \prec 0, \tag{6}$$

$$\mathrm{He}\left(D^\top P_g F\right) - \tau_0 \Pi_0 - \tau_5 \Pi_1 - \tau_4 \Pi_4 \prec 0, \tag{7}$$

where

$$\begin{aligned}
&D = \begin{bmatrix} A & B & -F_{nl}(v_{ref})B \end{bmatrix}, && F = \begin{bmatrix} I_2 & 0 & 0 \end{bmatrix}, \\
&F_C = \mu_C mg, && F_S = \mu_S mg, \\
&\pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, && \pi_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \\
&\pi_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, && \\
&\Pi_0 = \pi_4^\top \pi_4 - F^\top P_g F, && \Pi_1 = \pi_3^\top \pi_3 - F_S^2 \pi_4^\top \pi_4, \\
&\Pi_2 = F_C^2 \pi_4^\top \pi_4 - \pi_3^\top \pi_3, && \Pi_3 = -\mathrm{He}\left((\pi_1 + v_{ref}\pi_4)^\top \pi_3\right), \\
&\Pi_4 = (\pi_1 + v_{ref}\pi_4)^\top (\pi_1 + v_{ref}\pi_4), &&
\end{aligned}$$

then trajectories of system (5) *globally converge to the set*

$$\mathcal{A}_{v_{ref}}(P_g) = \left\{\varepsilon \in \mathbb{R}^2 \mid \varepsilon^\top P_g \varepsilon < 1\right\}.$$

Hence, the set $\mathcal{A}_{v_{ref}}(P_g)$ is a solution to Problem 1.

Proof : The proof of this theorem is based on the following Lyapunov function:

$$V_g(\varepsilon) = \varepsilon^\top P_g \varepsilon.$$

with $P_g = P_g^\top \succ 0$. Its time-derivative along the trajectories of (5) leads to:

$$\dot{V}_g(\varepsilon) = \xi_g^\top \left(D^\top P_g F + F^\top P_g D\right) \xi_g,$$

where $\xi_g = \mathrm{col}\left(\varepsilon, F_{nl}(\varepsilon_1 + v_{ref}), 1\right)$.

Inspired by [24], one wants to verify that there exists a class $\mathcal{K}$ function α such that $\dot{V}_g(\varepsilon) \leq -\alpha(V_g(\varepsilon))$, for all ε such that $\varepsilon^\top P_g \varepsilon \geq 1$ (i.e. for any $x \in \mathbb{R}^2 \backslash \mathcal{A}_{v_{ref}}(P_g)$), and for all nonlinearities F_{nl} satisfying Lemma 1. The existence of such a function α is discussed in two different cases.

- *First case: $\varepsilon_1 \neq -v_{ref}$.*

  Using the augmented vector ξ_g and the definition of the π_i's, the conditions of Lemma 1 read:

$$\begin{aligned}
\xi_g^\top (\pi_3^\top \pi_3 - F_S^2 \pi_4^\top \pi_4) \xi_g &\leq 0, \\
\xi_g^\top (F_C^2 \pi_4^\top \pi_4 - \pi_3^\top \pi_3) \xi_g &\leq 0, \\
-\xi_g \,\mathrm{He}\left((\pi_1 + v_{ref}\pi_4)^\top \pi_3\right) \xi_g &\leq 0.
\end{aligned}$$

Then, by using the compact notation Π_i's, inspired by the S-variable approach, define the following function:

$$\mathcal{L} \; = \dot{V}_g(\varepsilon) - \tau_0(1 - \varepsilon^\top P_g \varepsilon) - \tau_1 \xi_g^\top \Pi_1 \xi_g - \tau_2 \xi_g^\top \Pi_2 \xi_g - \tau_3 \xi_g^\top \Pi_3 \xi_g \tag{8}$$

for positive scalar τ_i, $i \in \{0, 1, 2, 3\}$.

Thus, a sufficient condition to ensure that $\mathcal{L}$ in (8) is negative-definite for $\varepsilon_1 \neq -v_{ref}$ is the existence of $\tau_0, \tau_1, \tau_2, \tau_3 \geq 0$ such that (6) is verified.

- *Second case:* $\varepsilon_1 = -v_{ref}$.

  Using Lemma 1 this time leads to:

$$\begin{aligned} \xi_g^\top (\pi_3^\top \pi_3 - F_S^2 \pi_4^\top \pi_4) \xi_g &\leq 0, \\ \xi_g^\top (\pi_1 + v_{ref} \pi_4)^\top (\pi_1 + v_{ref} \pi_4) \xi_g &\leq 0, \\ -\xi_g \, \mathrm{He} \left((\pi_1 + v_{ref} \pi_4)^\top \pi_3 \right) \xi_g &\leq 0. \end{aligned}$$

  Let us define $\mathcal{L}$ for $\varepsilon_1 = -v_{ref}$ as

$$\mathcal{L} \; = \dot{V}_g(\varepsilon) - \tau_0(1 - \varepsilon^\top P_g \varepsilon) - \tau_5 \xi_g^\top \Pi_1 \xi_g - \tau_4 \xi_g^\top \Pi_4 \xi_g \tag{9}$$

  for the same τ_0 as defined previously and positive scalars τ_4, τ_5.

  $\mathcal{L}$ in (9) is negative-definite on the set $\varepsilon_1 = -v_{ref}$ if there exist $\tau_0, \tau_4, \tau_5 \geq 0$ such that (7) is verified.

Consequently, if (6) and (7) are verified simultaneously then there exists $\alpha > 0$ such that $\mathcal{L} \; \leq -\alpha \varepsilon^\top \varepsilon$. By definition one gets $\dot{V}_g(\varepsilon) - \tau_0(1 - \varepsilon^\top P_g \varepsilon) \leq \mathcal{L}$ and since $\dot{V}_g(\varepsilon) \leq \dot{V}_g(\varepsilon) - \tau_0(1 - \varepsilon^\top P_g \varepsilon)$ on $\mathcal{A}_{v_{ref}}(P_g)$, it follows $\dot{V}_g(\varepsilon) \leq -\alpha \varepsilon^\top \varepsilon$ for any $x \in \mathbb{R}^2 \backslash \mathcal{A}_{v_{ref}}(P_g)$. That ends the proof. $\square$

Remark 2. *Relations* (6)-(7) *are quasi-LMIs since there is a nonlinearity due to the product between the scalar* τ_0 *and the matrix* P_g*. Note that a necessary condition to get* (6)-(7) *is that*

$$A^\top P_g + P_g A + \tau_0 P_g \prec 0.$$

As explored in [22], it means that A must be Hurwitz and the following inequality must hold:

$$\tau_0 \leq \tau_0^{max} = 2 \max_{\mu \text{ an eigenvalue of } A} (|\Re(\mu)|).$$

The previous remark emphasizes an efficient way to solve (6)-(7). A solution is to use a line-search for several given fixed values of $\tau_0 \in [0, \tau_0^{max}]$ and, in this case, (6)-(7) turn into LMIs.

To get an outer-estimate close to the real attractor, one can solve various optimization problems. For the minimization of the maximal axis, a solution is:

$$\begin{array}{ll} \min & -\eta \\ \text{subject to} & \text{(6) and (9) hold with } P_g \succeq \eta I_2. \end{array} \tag{10}$$

Remark 3. *Note that the analysis conducted here differs from the one in [22] where the sign function was not set-valued and the analysis was then not taking into account the behavior when $\varepsilon_1 = -v_{ref}$. Nevertheless, the lemmas derived in [22] still hold and for instance, one can show that the LMIs (6) and (7) are feasible as long as $k, k_v > 0$. That means that there always exists an attractor (eventually very large) for system (5).*

4. Regional Asymptotic Stability Analysis

Contrary to the previous section, this one concentrates on a regional analysis to solve Problem 2.

4.1. Stability of the linearized system

Similarly to the techniques used in presence of saturation or backlash nonlinearities (see, for example, [24, 25]), the fact that $\phi_{v_{ref}}$ is continuous with $\phi_{v_{ref}}(0) = 0$, is not sufficient to study the regional stability of the origin and we must consider the stability of its linearization around an equilibrium point to apply the first Lyapunov principle [23]. To this extend, we consider a rewriting of system (5) as follows:

$$\dot{\varepsilon}(t) = \left(A + B\Gamma_{v_{ref}} C\right) \varepsilon(t) + B \left(\phi_{v_{ref}}(\varepsilon_1(t)) - \Gamma_{v_{ref}} C\varepsilon(t)\right) \tag{11}$$

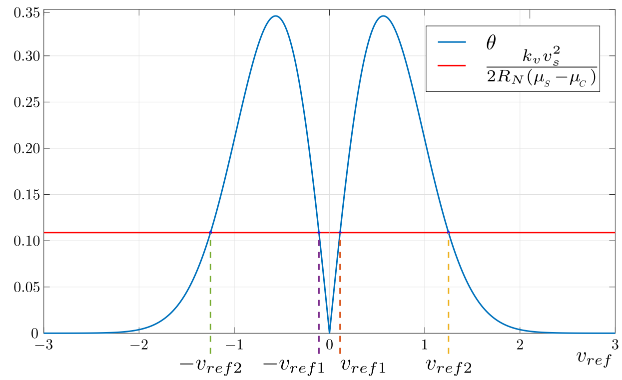


Figure 4: Nonlinear function g.

with A, B defined in (5), $C = [\,1\ 0\,]$ and

$$\Gamma_{v_{ref}} = \left.\frac{\partial \phi_{v_{ref}}}{\partial \varepsilon_1}\right|_{\varepsilon_1=0} = -2R_N(\mu_S - \mu_C)\frac{v_{ref}}{v_s^2}e^{-\frac{v_{ref}^2}{v_s^2}}.$$

System (11) reads:

$$\dot{\varepsilon}(t) = A_0\varepsilon(t) + B\psi_{v_{ref}} \tag{12}$$

with $A_0 = A + B\Gamma_{v_{ref}}C$ and $\psi_{v_{ref}} = \phi_{v_{ref}}(C\varepsilon) - \Gamma_{v_{ref}}C\varepsilon$. Consequently, the linearization of system (11) around the origin is:

$$\dot{\tilde{\varepsilon}}(t) = A_0\tilde{\varepsilon}(t) \tag{13}$$

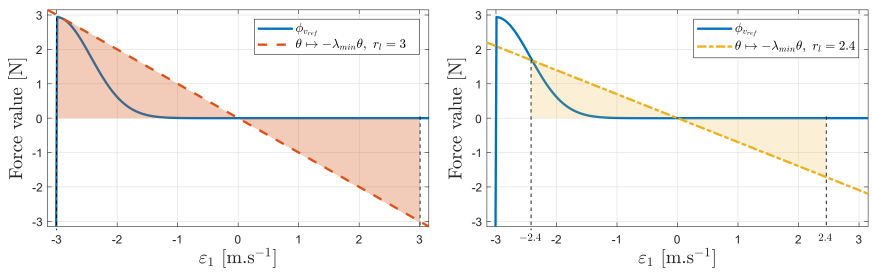
Hence, there exists a basin of attraction $\mathcal{D}_{v_{ref}}$ if and only if A_0 Hurwitz, which depends then on v_{ref}. The following lemma characterizes the stability of (13) as a function of v_{ref}.

Lemma 2. *Matrix A_0 is Hurwitz if and only if*

$$-k_v + 2R_N(\mu_S - \mu_C)\frac{v_{ref}}{v_s^2}e^{-\frac{v_{ref}^2}{v_s^2}} < 0,$$

or equivalently, $\theta(v_{ref}) = v_{ref}e^{-\frac{v_{ref}^2}{v_s^2}} < \frac{k_v v_s^2}{2R_N(\mu_S-\mu_C)}$.

We will now study the function θ defined in the previous lemma. Notice firstly that it is a nonlinear even function. Hence, looking at Figure 4, one



(a) $\lambda_{min} = 1.004,\ r_l = 3$

(b) $\lambda_{min} = 0.698,\ r_l = 2.4$

Figure 5: Sector conditions on the function $\phi_{v_{ref}}$ with $v_{ref} = 3$.

observes four roots such that $\theta(v_{ref}) = \frac{k_v v_s^2}{2R_N(\mu_S - \mu_C)}$. By denoting the two positive roots by v_{ref1} and $v_{ref2} > v_{ref1}$, one can exhibit several zones regarding the stability of A_0:

1. A_0 is Hurwitz for $v_{ref} \in (-\infty, -v_{ref2}) \cup (-v_{ref1}, 0) \cup (0, v_{ref1}) \cup (v_{ref2}, \infty)$ and, in virtue of the first Lyapunov principle [23], there exists a basin of attraction for (12) around the origin;
2. the origin of (13) is exponentially unstable for $v_{ref} \in (-v_{ref2}, -v_{ref1}) \cup (v_{ref1}, v_{ref2})$;
3. A_0 has poles on the imaginary axis for $v_{ref} = \pm v_{ref1}$ or $v_{ref} = \pm v_{ref2}$.

Consequently, without loss of generality, we assume here that $v_{ref} > 0$ and we concentrate on an inner-estimation of the basin of attraction around the origin for $v_{ref} \in (0, v_{ref1}) \cup (v_{ref2}, \infty)$.

4.2. Numerical inner-estimate of $\mathcal{D}_{v_{ref}}$

Note that $\phi_{v_{ref}}$ is bounded on $(-v_{ref}, \infty)$ and $\varepsilon_1 \phi_{v_{ref}}(\varepsilon_1) \le 0$ for $\varepsilon_1 \in (-v_{ref}, \infty)$, consequently, $\phi_{v_{ref}}$ is a local sector-bounded nonlinearity as depicted in Figure 5. Thus, the nonlinearity $\psi_{v_{ref}}$ satisfies the following Lemma, adapted from Lemma 1.6 in [25].

Lemma 3. *Given r_l such that $v_{ref} > r_l > 0$. For any $\varepsilon \in \mathcal{S}(r_l) = \{\varepsilon \in \mathbb{R}^2; -r_l \le C\varepsilon \le r_l\}$, the nonlinearity $\psi_{v_{ref}}$ satisfies the following:*

$$(\psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon)^\top (\psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon + \lambda C\varepsilon) \le 0. \tag{14}$$

for any positive scalar λ *satisfying*

$$\lambda \geq \lambda_{min}(r_l) > 0 \tag{15}$$

with

$$\lambda_{min}(r_l) = \max_{\varepsilon_1 \in [-r_l,0)\cup(0,r_l]} -\frac{\phi_{v_{ref}}(\varepsilon_1)}{\varepsilon_1} \geq -\Gamma_{v_{ref}}. \tag{16}$$

Proof : Recall that one gets $\phi_{v_{ref}}(\varepsilon_1) = \psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon$. For any $\varepsilon \in \mathcal{S}(r_l)$, by definition it follows:

$$\begin{cases} \phi_{v_{ref}}(\varepsilon_1) > 0, \frac{\partial \phi_{v_{ref}}}{\partial \varepsilon_1} < 0 & \text{if} \quad -r_l \leq \varepsilon_1 < 0 \\ \phi_{v_{ref}}(\varepsilon_1) = 0, \frac{\partial \phi_{v_{ref}}}{\partial \varepsilon_1} = \Gamma_{v_{ref}} & \text{if} \quad \varepsilon_1 = 0 \\ \phi_{v_{ref}}(\varepsilon_1) < 0, \frac{\partial \phi_{v_{ref}}}{\partial \varepsilon_1} < 0 & \text{if} \quad 0 < \varepsilon_1 \leq r_l \end{cases} \tag{17}$$

Hence, by taking the three cases of (17), one gets the following:

• Case 1: $-r_l \leq \varepsilon_1 < 0$. In this case the condition (14) holds provided that $\phi_{v_{ref}}(\varepsilon_1) + \lambda\varepsilon_1 \leq 0$, which is satisfied if $\lambda \geq \max_{\varepsilon_1 \in [-r_l,0)} -\frac{\phi_{v_{ref}}(\varepsilon_1)}{\varepsilon_1}$.

• Case 2: $\varepsilon_1 = 0$. In this case the condition (14) holds for any positive λ and therefore for any λ satisfying (15).

• Case 3: $0 \leq \varepsilon_1 < r_l$. In this case the condition (14) holds provided that $\phi_{v_{ref}}(\varepsilon_1) + \lambda\varepsilon_1 \geq 0$, which is satisfied if $\lambda \geq \max_{\varepsilon_1 \in (0,r_l]} -\frac{\phi_{v_{ref}}(\varepsilon_1)}{\varepsilon_1}$.

Then from these three cases, one can guarantee that there exists λ satisfying (15) such that (14) holds. Let us now focus on the way to calculate λ_{min} as described in (16). Define the function $g(\varepsilon_1) = -\frac{\phi_{v_{ref}}(\varepsilon_1)}{\varepsilon_1}$. To compute the maximal value of $g(\varepsilon_1)$ one studies its derivative in function of ε_1 on the interval $(-r_l, 0) \cup (0, r_l)$. More especially, one wants to find the values for which this derivative is equal to zero. By recalling that $\varepsilon_1 + v_{ref} > 0$, in this set, one gets

$$\begin{aligned} \frac{\partial g(\varepsilon_1)}{\partial \varepsilon_1} &= -\frac{\partial \phi_{v_{ref}}(\varepsilon_1)}{\partial \varepsilon_1}\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^2}\phi_{v_{ref}}(\varepsilon_1) \\ &= R_N(\mu_S - \mu_C)\left((\frac{1}{\varepsilon_1^2} + \frac{2(\varepsilon_1+v_{ref})}{\varepsilon_1 v_s^2})e^{-\frac{(\varepsilon_1+v_{ref})^2}{v_s^2}} - \frac{1}{\varepsilon_1^2}e^{-\frac{v_{ref}^2}{v_s^2}} \right) \end{aligned}$$

Therefore $\frac{\partial g(\varepsilon_1)}{\partial \varepsilon_1} = 0$ leads to $\phi_{v_{ref}}(\varepsilon_1) = \frac{\partial \phi_{v_{ref}}(\varepsilon_1)}{\partial \varepsilon_1}\varepsilon_1$, or equivalently, one has to find the solution to the equation:

$$e^{\frac{\varepsilon_1(\varepsilon_1+2v_{ref})}{v_s^2}} = 1 + \frac{2\varepsilon_1(\varepsilon_1 + v_{ref})}{v_s^2}, \tag{18}$$

Note that $\varepsilon_1 = 0$ is a trivial solution. To solve equation (18) for $\varepsilon_1 \neq 0$, one could use a dichotomy-kind of algorithm, by using the fact that $\phi_{v_{ref}}(\varepsilon_1)$ is monotonous for $\varepsilon_1 > -v_{ref}$. Hence, by denoting by v^* the solution to equation (18), one can deduce that

$$\lambda_{min}(r_l) = \begin{cases} -\frac{\partial \phi_{v_{ref}}}{\partial \varepsilon}(v^*), & \text{if } |v^* - v_{ref}| \leq r_l, \\ \max\left(-\frac{\phi_{v_{ref}}(r_l)}{r_l}, \frac{\phi_{v_{ref}}(-r_l)}{r_l}\right), & \text{otherwise.} \end{cases} \tag{19}$$

This is illustrated in Figure 5a for the first element $\left(-\frac{\partial \phi_{v_{ref}}}{\partial \varepsilon}(v^*)\right)$ and in Figure 5b for the second part $\left(\max\left(-\frac{\phi_{v_{ref}}(r_l)}{r_l}, \frac{\phi_{v_{ref}}(-r_l)}{r_l}\right)\right)$ where v^* is approximately -2.89.

Furthermore, one gets that

$$\lim_{r_l \to 0} \lambda_{min}(r_l) = -\lim_{r_l \to 0} \max\left(\frac{\phi_{v_{ref}}(r_l)}{r_l}, \frac{\phi_{v_{ref}}(-r_l)}{-r_l}\right) = -\Gamma_{v_{ref}}.$$

Since $\phi_{v_{ref}}$ is monotonous, as r_l decreases, $\lambda_{min}(r_l)$ gets closer to $-\Gamma_{v_{ref}}$.

Then the condition (16) is verified. □

Remark 4. *It is important to note that contrarily to the sector-condition for the dead-zone nonlinearity [25] where the slope of the nonlinearity at* 0 *is* 0*, in the current case the slope of the nonlinearity* $\phi_{v_{ref}}(\varepsilon)$ *at* 0 *is* $\Gamma_{v_{ref}} \neq 0$*. This is the reason for which in Lemma 3 there exists* $\lambda_{min}(r_l) \neq 0$*.*

By using Lemma 3, the following theorem proposes a solution to Problem 2.

Theorem 2. *Given* $v_{ref} \in (0, v_{ref1}) \cup (v_{ref2}, \infty)$*, if there exist* λ*,* $r_l \in (0, v_{ref})$*,* $P_l \in \mathbb{S}_+^2$ *and* $\tau > 0$ *such that the following matrix inequalities hold:*

$$\Phi(v_{ref}) \prec 0, \tag{20}$$

$$\begin{bmatrix} P_l & \star \\ C & r_l^2 \end{bmatrix} \succeq 0, \tag{21}$$

$$\lambda \geq \lambda_{min}(r_l) > 0, \tag{22}$$

where

$$\Phi(v_{ref}) = \begin{bmatrix} A_0^\top P_l + P_l A_0 - 2\tau\Gamma_{v_{ref}}(\Gamma_{v_{ref}} + \lambda)C^\top C & \star \\ B^\top P_l - \tau(2\Gamma_{v_{ref}} + \lambda)C & -2\tau \end{bmatrix},$$

then the origin of system (5) *is locally asymptotically stable and an estimation of the basin of attraction is:*

$$\mathcal{D}_{v_{ref}}(P_l) = \left\{\varepsilon \in \mathbb{R}^2 \mid \varepsilon^\top P_l \varepsilon \leq 1\right\}.$$

Hence, the set $\mathcal{D}_{v_{ref}}(P_l)$ is a solution to Problem 2.

Proof : Note first that the satisfaction of relation (21) is equivalent to $P_l \succeq C^\top C r_l^{-2}$. Consequently for any $\varepsilon \in \mathcal{D}_{v_{ref}}(P_l)$, we get $(C\varepsilon)^2 \leq r_l^2$ which implies that $\varepsilon \in \mathcal{S}(r_l)$ (see Lemma 3). Therefore, for any $\varepsilon_1 \in \mathcal{D}_{v_{ref}}(P_l)$, Lemma 3 applies and for any positive λ satisfying (22), condition (14) holds.

Consider the following Lyapunov function:

$$V_l(\varepsilon) = \varepsilon^\top P_l \varepsilon,$$

with $P_l \in \mathbb{S}_+^2$. Its time-derivative along system (12) leads to

$$\dot{V}_l(\varepsilon) = \varepsilon^\top (A_0^\top P_l + P_l A_0)\varepsilon + 2\varepsilon^\top P_l B \psi_{v_{ref}}.$$

By using Lemma 3 and the S-procedure, one gets for $\tau > 0$ and $\varepsilon \in \mathcal{S}(r_l)$:

$$\dot{V}_l(\varepsilon) \leq \dot{V}_l(\varepsilon) - 2\tau(\psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon)^\top (\psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon + \lambda C\varepsilon).$$

One wants to satisfy $\mathcal{L} = \dot{V}_l(\varepsilon) - 2\tau(\psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon)^\top (\psi_{v_{ref}} + \Gamma_{v_{ref}} C\varepsilon + \lambda C\varepsilon) < 0$. $\mathcal{L}$ reads $\mathcal{L} = \xi^\top \Phi(v_{ref})\xi$ where $\xi = \text{col}\left(\varepsilon, \psi_{v_{ref}}\right)$.

Hence, if relation (20) holds, it follows that $\mathcal{L} < 0$ and therefore $\dot{V}_l$ is definite-negative. If the conditions of the theorem are verified, then for any initial condition $\varepsilon(0) \in \mathcal{D}_{v_{ref}}(P_l)$, we get that $\varepsilon(t) \in \mathcal{D}_{v_{ref}}(P_l)$ and therefore the trajectories of system (5) converge to the origin. One can conclude that the origin of system (12) is regionally asymptotically stable in $\mathcal{D}_{v_{ref}}(P_l)$, or equivalently, (5) is regionally asymptotically stable in $\mathcal{D}_{v_{ref}}(P_l)$. $\square$

Remark 5. *From Lemma 3, one gets $\lambda > -\Gamma_{v_{ref}}$. Then, one can deduce that a necessary condition for the feasibility of* (20) *is that A_0 is Hurwitz, which holds since we consider $v_{ref} \in (0, v_{ref1}) \cup (v_{ref2}, \infty)$ (see Lemma 2).*

Regarding the feasibility of conditions (20)-(22), the following corollary can be stated.

Corollary 1.

1. *Given $v^\circ \geq v_{ref2}$, if there exist λ, $r_l \in (0, v_{ref})$, $P_l \in \mathbb{S}^2_+$ and $\tau > 0$ such that $\Phi(v^\circ) \prec 0$ and $\Phi(\infty) \prec 0$ together with* (21)-(22)*, then system* (5) *is locally asymptotically stable for any $v_{ref} \geq v^\circ$ with an inner-estimation of the basin of attraction $\mathcal{D}_v(P_l)$.*
2. *If A is Hurwitz, the matrix inequalities* (20)-(22) *are feasible.*

Proof :

First item: Let first note that the left-hand side in relation (20) can be rewritten as follows:

$$\begin{aligned}\Phi(v_{ref}) \quad = \quad & \begin{bmatrix} A^\top P_l + P_l A & \star \\ B^\top P_l - \tau\lambda C & -2\tau \end{bmatrix} + \Gamma_{v_{ref}} \begin{bmatrix} \mathrm{He}(P_l B C) - 2\tau\lambda C^\top C & \star \\ -\tau C & 0 \end{bmatrix} \\ & + \Gamma^2_{v_{ref}} \begin{bmatrix} -2\tau C^\top C & \star \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{23}$$

That shows that Φ is convex in $\Gamma_{v_{ref}}$. Furthermore, note that $\Gamma_{v_{ref}}$ is strictly decreasing with respect to v_{ref} and consequently, $\Phi(v_{ref})$ is convex in v_{ref} for $v_{ref} > v_{ref2}$ and that ends the proof for the first item.

Second item: The proof of the second item takes advantage of the proof of the first one. Given λ, r_l such that (22) holds, from (23), it follows:

$$\Phi(v_{ref}) = \Phi(\infty) + \Gamma_{v_{ref}} \,\mathrm{He}\left(\begin{bmatrix} P_l B - \tau(\Gamma_{v_{ref}} + \lambda) C^\top \\ -\tau \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix}\right) \tag{24}$$

Denote by $\mathcal{N}_1$ and $\mathcal{N}_2$ the basis of the Kernel of $\begin{bmatrix} C & 0 \end{bmatrix}$ and $\begin{bmatrix} B^\top P_l - \tau(\Gamma_{v_{ref}} + \lambda) C & -\tau \end{bmatrix}$, respectively. By using the Elimination Lemma, the satisfaction of $\Phi(v_{ref}) \prec 0$ is equivalent from (24) to:

$$\mathcal{N}_1^\top \Phi(\infty) \mathcal{N}_1 \prec 0, \qquad \mathcal{N}_2^\top \Phi(\infty) \mathcal{N}_2 \prec 0. \tag{25}$$

Furthermore, note that $\Phi(\infty)$ can be written as follows:

$$\Phi(\infty) = \begin{bmatrix} A^\top P_l + P_l A & 0 \\ 0 & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} P_l B - \tau\lambda C^\top \\ -\tau \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix}\right) \tag{26}$$

By using the Elimination Lemma, the satisfaction of $\Phi(\infty) \prec 0$ is equivalent to $A^\top P_l + P_l A \prec 0$. Hence, if A is Hurwitz, then there exists P_l such that $\Phi(\infty) \prec 0$ and consequently, (25) hold. Since $k \cdot P_l$ for $k \geq 1$ is also solution, there exists k large enough such that (21) holds. Therefore relations (20)-(22) also hold and the proof is completed. $\square$

Remark 6. *The first assertion of Corollary 1 implies that the basins of attraction have a minimal axis which is growing with v_{ref}.*

At this point, the aim is to derive an inner-estimate of the basin around the origin which is close to the real basin. To this extend, one can decide to solve the following optimization problem:

$$\begin{array}{cc} \min & \eta \\ \text{subject to} & (20)-(22) \text{ hold with } P_l \preceq \eta I_2. \end{array} \tag{27}$$

If r_l is a decision variable, this is not a semi-definite optimization problem since λ_{min} is nonlinearly related to r_l. Indeed, the relation (20) is an LMI provided that τ or λ is fixed. Moreover λ must satisfy condition (22), which depends on r_l. A way to have a set of LMI for (20) consists in fixing r_l, allowing to compute $\lambda_{min}(r_l)$, and to consider an additional variable $\gamma = \tau\lambda$ with the constraints $\gamma > \tau\lambda_{min}(r_l)$.

From this remark, we derive the following algorithm which aims at providing a sub-optimal solution:

1. Let $r_l = v_{ref}$, $\eta_{min} = \infty$ and $P_{min} = 0$;
2. Compute $\lambda_{min}(r_l)$ using (19);
3. Solve (27) using the previous paragraph:
    - if the problem is feasible, then

  - if $\eta < \eta_{min}$, then
    (a) $\eta_{min} \leftarrow \eta$ and $P_{min} \leftarrow P_l$;
    (b) decrease slightly r_l;
    (c) go back to step 2;
  - otherwise stop.

- if the problem is not feasible, then
  (a) decrease slightly r_l;
  (b) go back to step 2;

Due to the second assertion in Corollary 1, this algorithm will stop for r_l small enough. The result of this algorithm is a matrix P_{min} such that $\mathcal{D}_{v_{ref}}(P_{min})$ is a basin of attraction with a minimal axis which is a local maximum.

Combining the results of the previous parts leads to the main theorem of this article which is derived in the following section.

5. Global Asymptotic Stability Analysis

The main idea here is to use at the same time Theorems 2 and 1 to get a condition on v_{ref} in order to guarantee that the origin is globally asymptotically stable for system (5), and therefore to provide a solution to Problem 3.

Theorem 3. *Given $v_{ref} \in (0, v_{ref1}) \cup (v_{ref2}, \infty)$. If there exist λ, $r_l \in (0, v_{ref})$, τ, τ_0, τ_1, τ_2, τ_3, τ_4, τ_5 and $P_l, P_g \succ 0$ satisfying relations* (6)*,* (7)*,* (20)*,* (21) *and*

$$P_g - P_l \succeq 0 \tag{28}$$

then $\mathcal{A}_{v_{ref}}(P_g) \subseteq \mathcal{D}_{v_{ref}}(P_l)$ and the origin of (5) *is globally asymptotically stable.*

Proof : The satisfaction of relations (6), (7), (20), (21) means that Theorems 2 and 1 apply. It allows to characterize an attractor $\mathcal{A}_{v_{ref}}(P_g)$ towards which all the trajectories of system (5) converge and an estimation of the basin of attraction of the origin $\mathcal{D}_{v_{ref}}(P_l)$, in which the trajectories of (5) converges to the

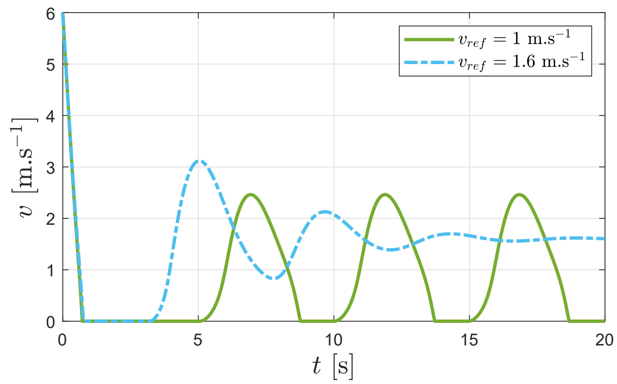


Figure 6: Numerical simulation of (4) for $v_{ref} \in \{1, 1.6\}$ m.s^{-1}, $v_0 = 6$ m.s^{-1} and $z_0 = 0$ m.

origin. Condition (28) guarantees the inclusion $\mathcal{A}_{v_{ref}}(P_g) \subseteq \mathcal{D}_{v_{ref}}(P_l)$. Then by definition of both sets, one can conclude that when the trajectories converging towards $\mathcal{A}_{v_{ref}}(P_g)$ enter in $\mathcal{D}_{v_{ref}}(P_l)$, they finally converge to the origin. That concludes the proof on the global asymptotic stability of the origin. $\square$

6. Numerical Simulations

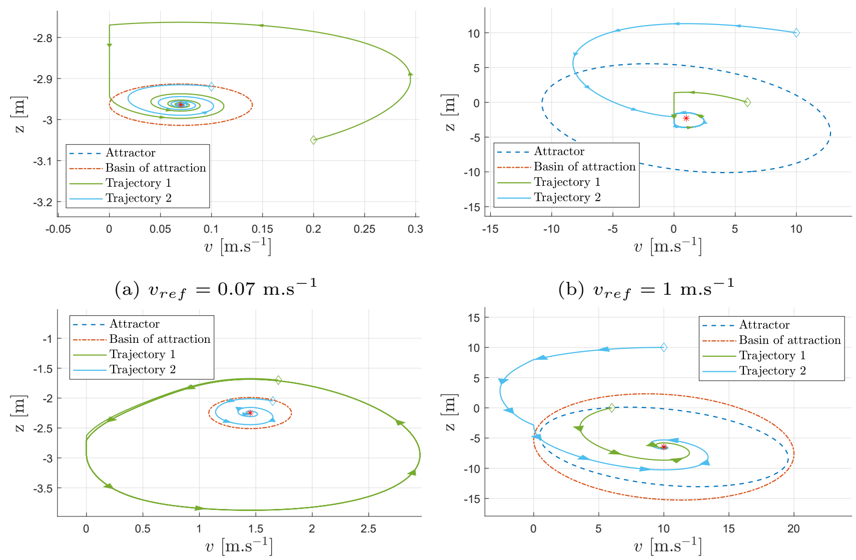
This section is dedicated to numerical results with the parameters defined as in Table 1.

Parameter	m	g	v_s	μ_C	μ_S	k	k_v
Value	1	9.81	0.8	0.2997	0.5994	2	1

Table 1: Parameters used for the simulations (taken from [26]).

The solver used for the LMIs is Mosek [27] together with YALMIP [28]. The simulations are conducted using a Newton-backward discretization scheme adapted for set-valued functions as proposed in [29].

Figure 6 depicts a numerical simulation of (4) with the parameters in Table 1 for two different reference speeds. The stick-slip phenomenon is clearly visible since after five seconds for $v_{ref} = 1$, a cycle of period $5s$ and of amplitude 2.4m.s^{-1} is emerging. For a reference speed higher than 1.45 m.s^{-1}, it



(a) $v_{ref} = 0.07$ m.s^{-1} (b) $v_{ref} = 1$ m.s^{-1}

(c) $v_{ref} = 1.45$ m.s^{-1} (d) $v_{ref} = 10$ m.s^{-1}

Figure 7: Phase portraits of (4) for four values of v_{ref}. The inner-approximation of $\mathcal{A}_{v_{ref}}$ is in dash-blue and the outer estimation of $\mathcal{D}_{v_{ref}}$ is in dash-dot-red. The equilibrium point is the red star and the two trajectories initiate from the diamond points.

seems that the stick-slip phenomenon disappears. In [2], the author are using a describing function analysis on a similar system, in our case, we would get that for $v_{ref} \geq 1.51$ m.s^{-1} there does not exist any limit cycle. This seems a very good estimation but it is not a rigorous analysis.

Using the methodology developed in this paper, one can compute the numerical values defining the interval of admissible v_{ref}, that is: $v_{ref1} = 0.11$ and $v_{ref2} = 1.25$. Then, for $v_{ref} \in (0, v_{ref1}) \cup (v_{ref2}, \infty)$, there exists a basin of attraction around the origin. Using Theorem 3, we find that for $v_{ref} \geq 9.59$, the equilibrium point is globally asymptotically stable[1]. As noted before, there always exists an attractor for the system but the existence or not of a basin of attraction leads to four scenarios:

[1]Using the method of [2], one gets a smaller value for global asymptotic stability. Nevertheless, it does not estimate the basin of attraction and the attractor.

1. $v_{ref} \in (0, 0.11)$: there exists a basin of attraction around the equilibrium point;
2. $v_{ref} \in [0.11, 1.25]$: the equilibrium point is not asymptotically stable;
3. $v_{ref} \in (1.25, 9.59)$: there exists a basin of attraction which does not include the attractor;
4. $v_{ref} \geq 9.59$: the basin of attraction contains the attractor and the equilibrium point is globally asymptotically stable.

Figure 7 shows the result of simulations in these four cases. The attractor[2] is computed with Theorem 1 and the basin of attraction with Theorem 2.

In Figure 7a, one can see that the trajectories are converging even if they start far away from the basin of attraction. Simulations tends to show that the equilibrium point is globally exponentially stable but our analysis does not reflect that.

In the second case (Figure 7b), there does not exist a basin of attraction. The attractor contains the oscillations but is very large compared to the real oscillations. The conservatism might come from the use of quadratic Lyapunov functions and a relatively rude encapsulation of the nonlinear friction term (see Figure 3).

Figure 7c shows that the inner-approximation of the basin of attraction is good and for a trajectory initiated close but outside this estimation, the stick-slip phenomenon occurs. The equilibrium point is not globally asymptotically stable but it stays locally asymptotically stable.

Finally, in the last case (Figure 7d), the attractor is included in the inner-approximation of the basin of attraction and the equilibrium point is globally asymptotically stable.

[2]Sometimes, the attractor is not displayed since it is not of the same order of magnitude as the basin of attraction.

7. Conclusion

This paper proposes three theorems dealing with the stability of a system subject to friction. The three proposed methods give a characterization of the global attractor of the system, an estimation of the basin of attraction of the origin when it is possible, and finally a global asymptotic stability condition. The conditions are given in terms of LMIs or quasi-LMIs[3] with efficient algorithms to solve them.

As depicted in the numerical simulations section, the reference speed for which global asymptotic stability exists is overestimated mainly because of the conservatism introduced in the global stability test. A first direction of research would be to estimate an attractor which might not be symmetrical and to study more precisely the friction function F_{nl} to get a better bounding. Another perspective would be to extend to higher order problems and real-life applications.

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