Design of a dynamical controller as a quasi-predictive control law for linear time-delay systems

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***Abstract*— A usual way to control time-delay systems is to design a predictive control law. Nevertheless, this theoretically appealing method is difficult to implement numerically in the sense that the implemented closed-loop system can be unstable. This paper paves the way of a new methodology to design a numerically safe approximated control law. It is based on the first quasi-Legendre polynomial coefficients of the control. Described and argued along the paper, several steps lead us to a dynamical controller which is easy to manipulate. After carrying on the design of this so-called quasi-predictive control law, we focused on its stability and robustness properties using fine criteria based on Bessel-Legendre inequality. At the end, by processing simulations and evaluating the performances of the controlled closed-loop on two examples, pros and cons of this new dynamical controller are pointed out.**

I. INTRODUCTION

Time-delays often occur during modeling of systems which process numerical or bio-physical information. Controlling such systems without taking into consideration the delay may induce degradation of performances and instability of the closed-loop. To bypass the problem, the so-called Smith predictor or dead time compensator was originally introduced in 1959. It consists in predicting the future of the state (see [1]) to design after a lag time a control law as if there is no delay. Rewritten as a backstepping method on the transport equation in [2], this nominal predictive control law is considered to be an appropriate solution (see [3]) to stabilize the system for any given delay under the condition of controllability of the system without delay.

Nevertheless, as presented in [4], when the infinite-dimensional part of this control law, modeled as an integral term, is realized by a dynamical system, the implementation can be unsafe. Indeed, as explained in [5], if the state matrix of the plant is not Hurwitz, it leads to an unstable pole-zero compensation in an internal loop. For these reasons, modifications are often added to the original Smith predictor to extend the stability properties to a larger range of systems (see [6,7]) or to reinforce the robustness with respect to uncertainties (see [8]). Then, as no guarantee of stability of the closed-loop system are proposed, the design of an implementable predictive control law is still an open problem.

Regarding the literature, many researchers have circumvented the problem by developing numerical approximations like rectangular or trapeze methods. Nevertheless, the closed-loop reveals itself to be a neutral time-delay system which contains an infinite number of characteristic roots in a vertical line (see [9,10]). Focusing on the case in which the line has a positive real part, the paper [11] pointed out a sufficient condition of instability of the system looped with the approximated control law. It means that, for a given stabilizing feedback gain, there exists a maximal allowable delay from which the implementation becomes unstable. These results have raised the question of the design of implementable controllers from the choice of the state feedback gain (e.g. feedback synthesis [12]) to the use of additional states (e.g. low-pass filter [13]).

In this paper, we proposed a new dynamical controller based on the first coefficients on Legendre polynomials of the integral term contained in the predictive control law. Then, the controller as a finite-dimensional model can be computed numerically and the closed-loop system seen as a time-delay system. Analyzing it, stability properties and robustness with respect to model uncertainties can be estimated with techniques developed before (mainly [14,15,16] dealing with Bessel-Legendre inequality). Two examples are tested at the end to illustrate that this new dynamical controller can be stable for a larger range of delays and more robust than other implementation methods.

Notations : Throughout the paper, $\mathbb{N}$ denotes the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^n$ the n dimensional Euclidian space and $\mathbb{R}^{n\times m}$ the set of $n \times m$ real matrices. For any square matrix $A \in \mathbb{R}^{n\times n}$, A^T represents its transpose, $\mathrm{tril}(A)$ its lower triangular part and $\mathcal{H}(A) = A + A^T$. Moreover, $\mathrm{diag}(d_0, \dots, d_N)$ is the diagonal matrix defined by its diagonal coefficients $(d_0, \dots, d_N)$ and the symbol $\otimes$ traduces a Kronecker product. Lastly, the set of square-integrable functions from $(-h, 0)$ to $\mathbb{R}^n$ is noted $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$ and the notation $u_t(\theta)$ stands for $u(t + \theta)$, for all $t \geq 0$ and all $\theta \in (-h, 0)$.

II. DESIGN OF THE DYNAMICAL CONTROLLER

In this section, the methodology to design the finite-dimensional dynamical controller is developed. The construction is based on the nominal predictive control law given by for instance Krstic in [2] which is in a second step approximated using a polynomial based approach. It consists of the truncation of the Fourier-Legendre serie of the input and the computation of the required first quasi-Legendre coefficients through a finite dynamical system. Then, our new dynamical controller can be seen as a quasi-predictive control

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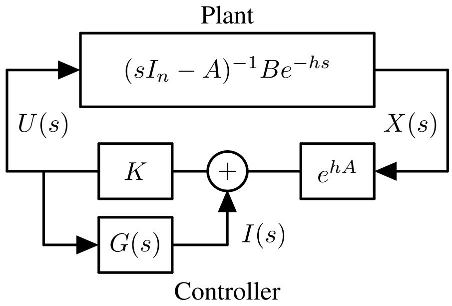


Fig. 1: Block diagram of the closed-loop system with the nominal controller.

law and its state representation which is easy to compute is obtained.

A. Nominal predictive control law

Let us consider the following linear system (1) subject to an input delay,

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, dynamical state matrix $A \in \mathbb{R}^{n\times n}$ and input matrix $B \in \mathbb{R}^{n\times m}$. The delay $h > 0$ is assumed to be constant but possibly uncertain.

The nominal control law (2) is a Volterra integral equation of the second kind.

$$u(t) = K\left(e^{hA}x(t) + \underbrace{\int_{-h}^{0} e^{-\theta A}Bu_t(\theta)d\theta}_{\iota(t)}\right). \tag{2}$$

As presented in [2], it ensures asymptotic stability of the origin $x = 0$ under the assumption $A + BK$ Hurwitz with $K \in \mathbb{R}^{m\times n}$ the control gain. However, as an infinite-dimensional law, the integral part $\iota(t)$ cannot be implemented directly.

In order to understand the induced problem, have a look at the closed-loop system in Laplace domain in Fig. 1. One inspects the transfer function $G(s)$ in between $U(s)$ and $I(s)$ (respectively Laplace transforms of $u(t)$ and $\iota(t)$) to obtain

$$G(s) = \int_{-h}^{0} e^{\theta(sI_n - A)}Bd\theta. \tag{3}$$

Considering its realization, one can use the most used Smith predictor form, $G(s) = (sI - A)^{-1}(I_n - e^{-h(sI_n-A)})B$, which turns out to be tricky to compute for unstable A because it introduces a pole-zero cancellation on each eigenvalue of A.

Focusing on numerical approximations [9,10], taking $G(s) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-\frac{k}{N}h(sI_n-A)}B$, other implementation issues appear. The controller needs a large amount of data and becomes unstable from a low maximal delay. But, in the following, A might be unstable and these limitations are exceeded.

Inspired by what has been proposed in the literature and motivated by the promising results of Legendre polynomials to give sufficient stability properties for time-delay systems as [14] suggests, one decided to approximate the term $\iota(t)$ by considering a truncated Fourier-Legendre decomposition of the signal u_t (i.e. working on its first Legendre polynomial coefficients).

B. Approximated control law on Legendre polynomials

Define the following approximated finite-dimensional control law

$$u(t) = K\left(e^{hA}x(t) + \iota_N(t)\right), \tag{4}$$

where the approximated integral term $\iota_N(t)$, defined by Equation (5), approximates $\iota(t)$ on Legendre polynomials as ruled on Theorem 1,

$$\iota_N(t) = \mathcal{C}_N U_N(t). \tag{5}$$

From one side, the vector $U_N(t) = \begin{bmatrix} u_0^T(t) & \dots & u_N^T(t)\end{bmatrix}^T$ collocates each Legendre coefficients from 0 to N. For each $k \in \mathbb{N}$, these coefficients $u_k(t) \in \mathbb{R}^m$, projection of the transported signal $u_t \in \mathcal{L}^2(-h, 0; \mathbb{R}^m)$ on the k-th Legendre polynomial L_k are recalled below

$$u_k(t) = \int_{-h}^{0} L_k(\theta)u_t(\theta)d\theta,$$

where,

$$L_k : \begin{cases} [-h, 0] \to \mathbb{R} \\ \qquad \theta \to (-1)^k \sum_{l=0}^{k} (-1)^l \binom{k}{l}\binom{k+l}{l}\left(\frac{\theta+h}{h}\right)^l \end{cases},$$

taking the notation $\binom{k}{l}$ for the binomial coefficient.
From the other side, matrix $\mathcal{C}_N$ gathers each associated weighting coefficients and is defined by

$$\begin{cases} \Gamma_k = \int_{-h}^{0} e^{-\theta A}L_k(\theta)d\theta, \ \forall k \in \mathbb{N}, \\ \mathbf{I}_N = \mathrm{diag}(1, \dots, 2N+1), \\ \mathcal{C}_N = \begin{bmatrix}\Gamma_0 & \dots & \Gamma_N\end{bmatrix}\left(\frac{1}{h}\mathbf{I}_N \otimes B\right). \end{cases}$$

Remark 1: For computational issues, in the case A is non-singular, the matrices sequence $(\Gamma_k)_{k\in\mathbb{N}}$ can be calculated recursively using Legendre polynomials properties of [17].

$$\begin{cases} \Gamma_0 &= -A^{-1}(I_n - e^{hA}), \\ \Gamma_1 &= A^{-1}(I_n + e^{hA}) - \frac{2}{h}\Gamma_0, \\ \Gamma_{k+1} &= \Gamma_{k-1} - \frac{2(2k+1)}{h}A^{-1}\Gamma_k. \end{cases}$$

Indeed, a recursion relation on Legendre polynomials gives for all $k \geq 2$, $L'_{k+1}(\theta) - L'_{k-1}(\theta) = \frac{2(2k+1)}{h}L_k(\theta)$.

Remark 2: One can also note that if A is nilpotent at order N^*, then for each $k > N^*$, $\Gamma_k = 0_n$ and the approximation ι_N at order $N = N^*$ gives the exact value of $\iota(t)$. It means that $N^* + 1$ Legendre coefficients are enough to compute the integral term.

The following theorem ensures that the simple convergence from $\iota_N(t)$ to $\iota(t)$ holds.

Theorem 1: The approximated integral $\iota_N(t)$ converges to the expected integral $\iota(t)$.

Proof: From the definition of $\mathcal{C}_N$, the error $\tilde{\iota}_N(t)$ between $\iota(t)$ and $\iota_N(t)$ is equal to

$$\tilde{\iota}_N(t) = \int_{-h}^{0} e^{-\theta A} B \left(u_t(\theta) - \sum_{k=0}^{N} \frac{2k+1}{h} L_k(\theta) u_k(t) d\theta \right).$$

Hence, using Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\tilde{\iota}_N(t)|^2 &= |\int_{-h}^{0} e^{-\theta A} B e_N(t,\theta) d\theta|^2, \\ &\leq h \int_{-h}^{0} e_N^T(t,\theta) B^T e^{-\theta(A^T+A)} B e_N(t,\theta) d\theta, \end{aligned}$$

where $e_N(t,.)$ is the Fourier-Legendre error of the signal u_t on the orthogonal basis $(L_k)_{k\in\mathbb{N}}$ of $\mathcal{L}^2(-h,0;\mathbb{R}^m)$ defined as

$$e_N(t,\theta) = u_t(\theta) - \sum_{k=0}^{N} \frac{2k+1}{h} L_k(\theta) u_k(t).$$

Furthermore, there exists $\alpha > 0$ such that $B^T e^{-\theta(A^T+A)} B \leq \alpha I_m$, for all $\theta \in (-h, 0)$. Then,

$$|\tilde{\iota}_N(t)|^2 \leq h\alpha \int_{-h}^{0} e_N^T(t,\theta) e_N(t,\theta) d\theta.$$

Assuming $u_t \in \mathcal{L}^2(-h,0;\mathbb{R}^m)$, the $\mathcal{L}^2$-norm convergence of Fourier-Legendre error $e_N(t,.)$ concludes the proof. ■

This theorem justifies that $\iota_N(t)$ is a well-chosen term to approximate $\iota(t)$. Therefore, $u(t)$ defined by Equation (4) remains close to the nominal one described by Equation (2). Otherwise, this result just carries the problem of implementation over to realize U_N. Indeed, besides having an accurate approximant of the integral term, the own dynamics of Legendre coefficients $U_N(t)$ described by Equation (6) cannot be implemented safely. Indeed, by derivation, it represents N integrators in cascade and makes appear $u(t-h)$.

$$\begin{cases} \dot{U}_N(t) = h^{-1}\mathcal{A}_N U_N(t) + \mathcal{B}_N u(t) - \mathcal{B}_N^* e_N(t,-h), \\ e_N(t,-h) = u(t-h) - h^{-1}\mathcal{C}_N^* U_N(t), \end{cases} \tag{6}$$

with

$$\begin{cases} \mathbf{1}_N = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T, \ \mathbf{1}_N^* = \begin{bmatrix} (-1)^0 & \dots & (-1)^N \end{bmatrix}^T, \\ \mathbf{L}_N = \text{tril}(\mathbf{1}_N \mathbf{1}_N^T - \mathbf{1}_N^* \mathbf{1}_N^{*T}), \\ \mathcal{A}_N = -\left((\mathbf{L}_N + \mathbf{1}_N^* \mathbf{1}_N^{*T}) \mathbf{I}_N \right) \otimes I_m, \\ \mathcal{B}_N = \mathbf{1}_N \otimes I_m, \ \mathcal{B}_N^* = \mathbf{1}_N^* \otimes I_m, \ \mathcal{C}_N^* = (\mathbf{1}_N^{*T} \mathbf{I}_N) \otimes I_m. \end{cases}$$

The proof of this dynamical equation can be found in [16] and relies on integration by parts and Legendre properties at the boundaries.

Then, a last modification is needed to construct a safe implementable dynamical controller.

C. Quasi-predictive control law and its state representation

This last step is made up of the construction of a stable model close to system (6). In previous work [16], putting aside the error $e_N(t,-h)$ done at order N on $u(t-h)$, it is proven that the finite-dimensional model in between $u(t)$ and $u(t-h)$ obtained is a Pade approximant of the single delay h (i.e. the function e^{-hs} in Laplace domain). Moreover, having a look to numerical approximation of time-delay systems on Legendre polynomials we found that such techniques are also used to compute characteristic roots as accurate as desired (see [18]). Following this idea, removing this model error, the dynamical controller of state $X_c \in \mathbb{R}^{m(N+1)}$ described by Equation (7) is finally proposed.

$$\begin{cases} \dot{X}_c(t) = h^{-1}\mathcal{A}_N X_c(t) + \mathcal{B}_N u(t), \\ u(t) = K\left(e^{hA}x(t) + \mathcal{C}_N X_c(t)\right), \end{cases} \tag{7}$$

Instead of realizing the transfer function $G(s)$ or approximating it by delayed samples [9,10,11], we decided to work on a transfer function $G_N(s)$ associated to the state representation $\begin{pmatrix} h^{-1}\mathcal{A}_N & \mathcal{B}_N \\ \mathcal{C}_N & 0 \end{pmatrix}$.

Remark 3: One can remark that, if $h \to 0$, then $u(t) \to Kx(t)$. Indeed, the state $X_c(t)$ converges quickly to zero and, by Taylor expansions, we have $\mathcal{C}_N \underset{h\to 0}{\longrightarrow} \begin{bmatrix} B & 0 & \dots & 0 \end{bmatrix}$.

To sum up, the input $u(t)$ is given by the nominal control law (2) where the integral term $\iota(t)$ is approximated by Equation (5) and where Legendre coefficients $U_N(t)$ are replaced by quasi-Legendre coefficients $X_c(t)$ which follows a dynamics chosen arbitrary close to the one of $U_N(t)$. Thus, this new finite-dimensional dynamical controller, constructed smartly, is then implementable and can be seen as a quasi-predictive control law. It is a finite-dimensional model with an input $x(t)$ and an output $u(t)$ and of state representation $\begin{pmatrix} h^{-1}\mathcal{A}_N + \mathcal{B}_N K \mathcal{C}_N & \mathcal{B}_N K e^{hA} \\ K\mathcal{C}_N & K e^{hA} \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}$.

In the next paragraph, the stability and robustness of the closed-loop system with our finite-dimensional dynamical controller is investigated.

III. STABILITY AND ROBUSTNESS OF THE CLOSED-LOOP SYSTEM

In this section, one focuses on stability and robustness properties of our closed-loop system and compares it with other time-delay compensators.

A. Closed-loop system

The design of a finite-dimensional dynamical controller leads to a closed-loop system which can be modeled as a time-delay system.

Indeed, the closed-loop system verifies Equation (8).

$$\xi(t) = \mathbb{A}\xi(t) + \mathbb{B}_d \mathbb{C}_d \xi(t-h), \tag{8}$$

where the state $\xi(t) = \begin{bmatrix} x(t) \\ X_c(t) \end{bmatrix}$ belongs to $\mathbb{R}^\sigma$ with $\sigma = n + m(N+1)$ and matrices $\mathbb{A}$, $\mathbb{B}_d$, $\mathbb{C}_d$ defined for given matrices A, B, K and constant delay h by

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ B_c & A_c \end{bmatrix}, \ \mathbb{B}_d = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \mathbb{C}_d = \begin{bmatrix} D_c & C_c \end{bmatrix}.$$

The characteristic roots of $sI_\sigma - \mathbb{A} + \mathbb{A}_d e^{-hs}$ can directly be estimated through an algorithm based on Tchebychev discretization developed and explained in [19]. A spectral analysis is possible giving an idea of the performances and the stability with an error margin made by discretization. To go further into details, thanks to a Lyapunov-Krasovskii functional, a fine enough sufficient condition of stability and robustness with respect to the delay can be given.

B. Sufficient condition of stability

With similar calculation as presented in [14,15], a rewriting of the hierarchical sufficient condition of stability can be given at an order $M \in \mathbb{N}$ as it was already done in [16].

Theorem 2: If there exist symmetric positive definite matrices $P \in \mathbb{S}^{\sigma+m(M+1)}$, $S \in \mathbb{S}^m$, $R \in \mathbb{S}^m$ and a delay h such that

$$\begin{bmatrix} \mathcal{H}(E(h)PF) + \tilde{S} + \tilde{R} & E(h)PG \\ G^T P^T E(h)^T & -S \end{bmatrix} < 0, \qquad (9)$$

where M is the order of the linear matrix inequality and

$$\begin{cases} E(h) = \mathrm{diag}(I_\sigma, hI_{m(M+1)}), \\ F = \begin{bmatrix} \mathbb{A} & \mathbb{B}_d \mathcal{C}_M^* \\ \mathcal{B}_M \mathbb{C}_d & \mathcal{A}_M \end{bmatrix}, \; G = \begin{bmatrix} \mathbb{B}_d \\ -\mathcal{C}_M^* \end{bmatrix}, \\ H = \begin{bmatrix} \mathbb{C}_d & -\mathbf{1}_M^T \mathbf{I}_M \otimes I_m \end{bmatrix}, \; \tilde{S} = H^T S H, \\ \tilde{R} = \begin{bmatrix} h\mathbb{C}_d^T R \mathbb{C}_d & 0 \\ 0 & -h\mathbf{I}_M \otimes R \end{bmatrix}, \end{cases}$$

then system (8) is asymptotically stable, for the delay h.

Proof: Following the proof given in [16], consider the Lyapunov-Krasovskii functional candidate V given by (10).

$$V(t,\xi) = \begin{bmatrix} \xi(t) \\ \xi_M(t) \end{bmatrix}^T P \begin{bmatrix} \xi(t) \\ \xi_M(t) \end{bmatrix} + V_S(t,\xi) + V_R(t,\xi), \quad (10)$$

with

$$\begin{cases} \xi(t) = \begin{bmatrix} X(t) \\ X_c(t) \end{bmatrix}, \; \xi_M(t) = \begin{bmatrix} \int_{-h}^{0} \mathbb{C}_d \xi(t+\tau) L_0(\tau) d\tau \\ \cdots \\ \int_{-h}^{0} \mathbb{C}_d \xi(t+\tau) L_M(\tau) d\tau \end{bmatrix}, \\ V_S(t,\xi) = \int_{-h}^{0} \big(\mathbb{C}_d \xi(t+\tau)\big)^T S \big(\mathbb{C}_d \xi(t+\tau)\big) d\tau \\ \qquad - \xi_M^T(t) (\frac{1}{h} \mathbf{I}_M \otimes S) \xi_M(t), \\ V_R(t,\xi) = \int_{-h}^{0} (h+\tau) \big(\mathbb{C}_d \xi(t+\tau)\big)^T R \big(\mathbb{C}_d \xi(t+\tau)\big) d\tau, \end{cases}$$

The positivity of V is ensured by the positive definiteness of P, S and R and application of Bessel lemma on V_S.
The derivative of V along the trajectories of system (8) is bounded from above by

$$\begin{bmatrix} \xi(t) \\ \frac{1}{h}\xi_M(t) \\ e_M(t) \end{bmatrix}^T \begin{bmatrix} \mathcal{H}(E(h)PF) + \tilde{S} + \tilde{R} & E(h)PG \\ G^T P^T E(h)^T & -S \end{bmatrix} \begin{bmatrix} \xi(t) \\ \frac{1}{h}\xi_M(t) \\ e_M(t) \end{bmatrix},$$

with the error $e_M(t) = \mathbb{C}_d \xi(t-h) - \mathcal{C}_M^* \xi_M(t)$.
Calculation details relied on application of Bessel lemma can be found in [16].
Therefore, if the linear matrix inequality (9) is satisfied, the origin of system (8) with that delay h is asymptotically stable by application of the Lyapunov-Krasovskii theorem. ■

Remark 4: Notice that following [14], the principle of hierarchy applied and if inequality (9) is satisfied for a delay h at order M^*, then this inequality is also verified at order $M > M^*$.

Remark 5: According to Remark 3, since $A+BK$ is Hurwitz and because the linear matrix inequality is continuous in h, we expect to find an interval of delay which assert stability from 0 to a maximal bound.

Lastly, assuming that the closed-loop system is stable for a nominal delay h, one proposes a theorem which asserts the stability of the system (8) for some delays around.

C. Robustness with respect to delay uncertainties

Consider an unknown delay h belonging to the interval $[h - \delta h, h + \delta h]$. A criterion of robusteness is required to ensure stability of the whole system with a controller implemented for a delay h for all constant delays in the pocket $[h - \delta h, h + \delta h]$.

Thanks to the previous Lyapunov-Krasovskii approach, stability results can be extended to robustness. One defines the matrix $\phi(h)$ which appears in the linear matrix inequality (9) for a given h.

$$\phi(h) = \begin{bmatrix} \mathcal{H}(E(h)PF) + \tilde{S} + \tilde{R} & E(h)PG \\ G^T P^T E(h)^T & -S \end{bmatrix}$$

It is clear that $\phi(h)$ is affine in h. From this statement, it is possible to rewrite the theorem proposed in [15]. The following revisited theorem gives a linear matrix inequality which ensures stability for all constant delays in the pocket $[h^-, h^+]$.

Theorem 3: If there exist symmetric positive definite matrices $P \in \mathbb{S}^{\sigma+m(M+1)}$, $S \in \mathbb{S}^m$ and $R \in \mathbb{S}^m$ and two positive scalars h^- and h^+ such that

$$\phi(h^-) < 0, \; \phi(h^+) < 0, \qquad (11)$$

then system (8) is asymptotically stable, for all delays in the interval $[h^-, h^+]$.

Proof: Consider the Lyapunov-Krasovskii functional candidate V given by (10). The positivity of V is again verified by the positive definiteness of P, S and R.
Then, from the linearity of $\phi(h)$ in h, the derivative of V for system (8) leads to

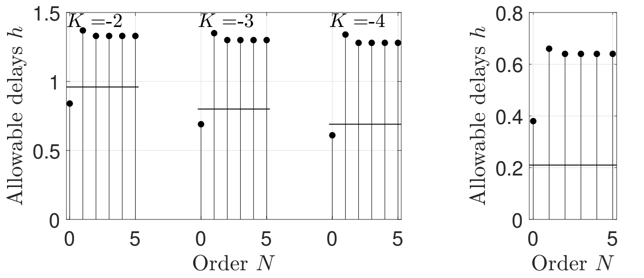
$$\dot{V}(t,\xi) \leq \phi(h) = \frac{h - h^-}{h^+ - h^-}\phi(h^+) + \frac{h^+ - h}{h^+ - h^-}\phi(h^-)$$

Thus, the negativity of $\phi(h^-)$ and $\phi(h^+)$ guarantees that the origin of system (8) is asymptotically stable for any constant delay $h \in [h^-, h^+]$. ■

Note that similar results could be found for robustness with respect to a time-varying delay which is an important issue (see [20]) or to model uncertainties on state matrix A or input matrix B.

IV. EXAMPLES

In this last section, the stability and robustness of our dynamical controller on two classical examples are evaluated. These simulations are conducted on purpose on unstable



(a) First example (b) Second example

Fig. 2: Maximal allowable delay given by Theorem 2.

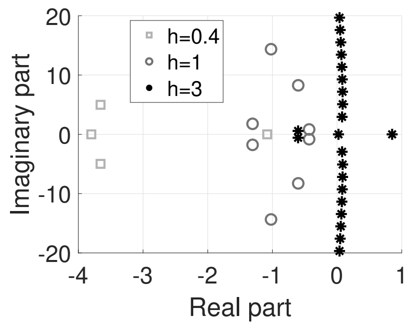


Fig. 3: Characteristic roots of the controlled system ($N = 1$).

systems because if A was stable, it would be better to set up usual tools [7]. From the examples, to get closer to the entire set of stability of our closed-loop system, a criterion of stability based on Bessel-Legendre as Theorem 2 is required and more precisely at an order $M \geq N$ because the design of our dynamical control is related to the $N + 1$ first Legendre coefficients. On Fig. 2, using Theorem 2 at order $M = 5$, the pointwise maximum allowable delay is given for each example and for a controller at order $N \in \{0, \ldots, 5\}$. It is compared with the delay from which classical rectangular or trapeze approximation methods fail to compute the nominal predictive control law (see the horizontal line). It is the one given by the stability of the nominal controller itself (i.e. $\det(I_m - KG(s))$ has at least one zero with positive real part) which is a necessary condition of stability of the closed-loop according to [11].

In the sense that for stability, the impact of K turns out to be minor, we decided to fix K for desired properties.

A. First example

Consider a 1-dimensional system (12), widely studied and known to be practically unstable by usual approximations of the nominal control law with $K = -2$ since $h > 0.405$ and to have no stabilizing feedback gain K since $h > 0.693$ as mentioned in [21].

$$\dot{x}(t) = x(t) + u(t - h). \tag{12}$$

First, taking $K = -2$, an estimation of the spectrum assignment of the closed-loop system with the dynamical controller at order $N = 1$ is computed for $h = 0.4$, $h = 1$ and $h = 3$ (see Fig. 3). The closed-loop system is stable for a larger bound of delays in comparison with approximation techniques. Indeed, thanks to Theorem 2 at order $M = 5$ and increasing pointwise the delay h, it appears that the system with our finite-dimensional dynamical controller is asymptotically stable for $0 < h < 1.3$ from order $N = 1$ (see Fig.2a).

Furthermore, for a delay equal to 0.4, 1 and 3 the eigenvalues have respectively positive real parts lower than -1, 0 and higher than 0. It means that there is a first interval of delay in which the performance given by $A+BK$ are conserved and a second interval in which the closed-loop stays stable but the performances are degraded. This is

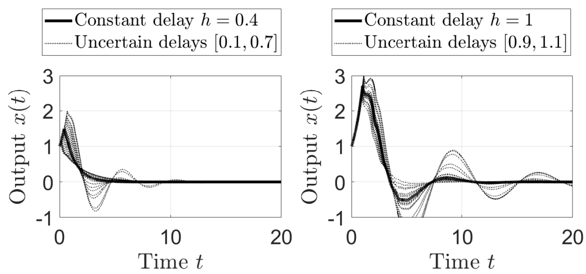


Fig. 4: First example : Simulation in time with controller at order $N = 1$.

confirmed by simulation in time of our closed-loop system with the dynamical controller (4) with $K = -2$ for the corresponding delays. This is highlighted by Fig. 4 where the output $x(t)$ is drawn in continuous line. The system with initial value $x(0) = 1$ is converging to the origin $x = 0$ for $h \in \{0.4, 1\}$ even if for $h = 1$ the exponential decay is reduced. Notice that, by rectangular approximation method of the integral, the output would diverge for $h = 1$.

Finally, Theorem 3 until order $M = 5$ gives a criterion of robustness with respect to delay uncertainties. For a controller at order $N = 1$ designed respectively for $h = 0.4$ and $h = 1$, the closed-loop system is proven to be stable on the pocket $[h - \delta h, h + \delta h]$ with $\delta h = 0.3$ and $\delta h = 0.1$. On Fig. 4, dotted lines represent the output of the controlled system with an error done on the delay randomly chosen on $[-\delta h, +\delta h]$. One can see that our closed-loop system at order $N = 1$ is robust with respect to delay uncertainties. Moreover, once the closed-loop system is stable for h, the performances are deteriorating while the error is increasing. We also noticed that the bound δh is independent of the order N. Same statements could be done with respect to uncertainties on the state or input matrix.

B. Second example

Consider now system (1) with $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and with a gain K designed using a LQR (linear quadratic regulator) method as in [2]. Notice that A is unstable.

The stability margin with respect to a constant delay h is computed with Theorem 2 for controllers from order $N = 0$ to $N = 5$ on Fig. 2b. The system remains stable until

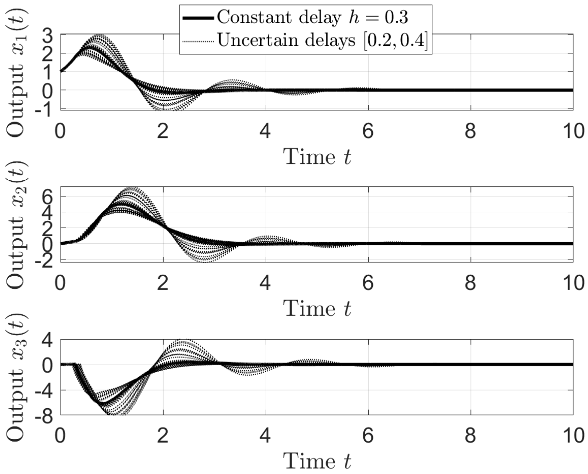


Fig. 5: Second example : Simulation in time with controller at order $N = 1$.

$h = 0.64$ for $N \geq 1$ which means a bound much larger than the one usually computed ($h = 0.3$). It is significant to note that $N = 1$ is enough to reach the maximum allowable delay because it also means that a smaller finite-dimensional set than classical approximation techniques (where hundreds of samples are taken into consideration) are required to design the controller. But, to balance these promising results, even if this dynamical controller is easy to compute and enlarges the interval of stability compared to the one reached by rectangular approximations, the stability of the closed-loop is still not guaranteed for all delays.

Finally, as for the previous example, Theorem 5 ensures some robustness with respect to uncertainties on the delay h. On Fig. 5, for a dynamical controller designed for $h = 0.3$ at order $N = 1$, the behavior of systems having a delay equal to 0.3 (continuous line) and 30 randomly chosen delays in 0.3 ± 0.1 (dotted lines) are drawn. The simulation in time confirms that the outputs $x(t)$ is converging from $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ to $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and that one more time we get the worst performances for the upper bound $h + \delta h$.

V. CONCLUSIONS

A finite-dimensional dynamical controller has been designed through several steps based on an interpolation of the nominal control law by first Legendre polynomials. Techniques based on Legendre polynomials has been then proposed to ensure its stability with respect to the delay and its robustness as well for uncertain delays. As in many papers in the literature, one obtains also a closed loop system which is asymptotically stable up to a prescribed upper bound.

Keeping the same feedback state representation, future work will focus on the choice of the control gain to optimize the closed-loop performances and robustness properties. In contrast to this late lumping approach, an early lumping approach could also be investigated.

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