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Abstract

By taking advantages of properties of the Laguerre polynomials, we propose a new inequality called Bessel-Laguerre integral inequality, which can be applied to stability analysis of linear systems with infinite distributed delays and with general kernels. The matrix corresponding to the system without the delayed term or the matrix corresponding to the system with the zero-delay is not necessarily assumed to be non-Hurwitz. Through a Laguerre polynomials approximation of kernels, the advantage of the method is that the original system is not needed to be transformed into an augmented one. Instead, it is represented as a system with additional signals that are captured by the Bessel-Laguerre integral inequality. Then, we derive a set of sufficient stability conditions that is parameterized by the degree of the polynomials. The particular case of gamma kernel functions can be easily considered in this analysis. Numerical examples illustrate the potential improvements achieved by the presented conditions with increasing the degree of the polynomial, but at the price of numerical complexity.

Key words: Systems with infinite distributed delays, Bessel-Laguerre integral inequality, Lyapunov method.

1 Introduction

Systems with distributed delays have been extensively studied in the literature, see e.g., [1], [2], [3], [4] and the references therein. Most of them were mainly focused on the class of finite distributed delays. In fact, infinite distributed delays appear in a wide range of applications, such as in population dynamics, in traffic flow dynamics of transportation systems, in machine tool vibration problem, in predator-prey model, see e.g., [5], [6], [7] and the references therein. Particularly, it was shown in [8] that gamma-distributed infinite delays with a gap can be encountered in the control over communication networks.

The analytical stability region of the traffic flow dynamics was presented in [9] using a frequency domain approach. Furthermore, necessary and sufficient stability condition was found in [6] for linear systems with gamma-distributed delays. Then consensus problems for a class of linear systems with gamma-distributed delays, with application to traffic flow dynamics, were analyzed in [5]. Note that in the traffic flow model on the ring considered in the above works, the matrix corresponding to the system without the delayed term is zero and the matrix corresponding to the system with the zero-delay has a zero eigenvalue. Thus, the infinite distributed delays in the traffic flow model on the ring have stabilizing effects.

To assess stability of systems with gamma-distributed infinite delays, a Lyapunov-based method was firstly provided in [10] by virtue of two kinds of integral inequalities with infinite intervals of integration. Furthermore, an efficient condition was proposed in [11] by a generalized integral inequality and its double integral extension. It is worth noting that in [10] and [11], if the matrix corresponding to the system without the delayed term or the matrix corresponding to the system with the zero-delay is not Hurwitz, the original system needs to be transformed into an augmented one, which may induce undesired additional dynamics and consequently some possible conservatism.

On the other side, based on Legendre orthogonal polynomials and Bessel inequality, a new set of integral inequalities that encompasses Jensen [12] and Wirtinger-

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based [13] inequalities as particular cases, was recently proposed to analyze the stability of systems with finite delays, see e.g, [14] and [15] for constant discrete delays, [16] for distributed delays, [17] for time-varying delays. The features of these inequalities are integrated into the construction of new Lyapunov functionals, leading to highly efficient stability criteria in terms of conservatism and complexity. It is worth noting that the Legendre polynomials are restricted to finite delays.

In the present paper, our objective is to derive less conservative stability conditions for linear systems with infinite distributed delays and with general kernels. Thanks to properties of the Laguerre polynomials, we first propose a new inequality called Bessel-Laguerre integral inequality. It is shown that the Bessel inequality provided in [15], [16] and [17] is not only relevant to Legendre polynomials with a particular inner product over a finite interval but also to Laguerre polynomials with an inner product over an infinite interval. Through a Laguerre polynomials approximation of kernels, it is not mandatory to transform the original system into an augmented one done in [10] and [11]. Instead, the original system is represented as a system with additional signals that can be captured by the newly introduced Bessel-Laguerre integral inequality. Then, an appropriate Lyapunov functional leads to a set of linear matrix inequalities (LMIs) that depends on the degree N of the Laguerre polynomial. The proposed method can easily include the case of gamma kernel functions. Two illustrative examples, including the traffic flow model on the ring, are given to show the potential improvements of the methodology, especially if one increases N, but at the price of additional decision variables and higher order of LMIs.

The structure of this paper is as follows. In Section 2, the system description with general kernel function is addressed, with the properties of the Laguerre polynomials and kernel approximation. The main results are presented in Sections 3 and 4, which include the newly introduced Bessel-Laguerre integral inequality and its application to stability analysis of infinite distributed delay systems with general kernels, respectively. Two illustrative examples are discussed in Section 5 and a brief summary in Section 6 concludes the paper.

Notations: The symbols $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}^+$ and $\mathbb{N}$ denote the set of real numbers, non-negative real numbers, non-negative integers and positive integers, respectively. $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P\succ 0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. For $x:\mathbb{R}\to\mathbb{R}^n$, we denote $x_t(\theta)\overset{\Delta}{=}x(t+\theta)$, $\theta\in[-h,0]$. The superscript 'T' stands for matrix transposition. Moreover, for any square matrix $A\in\mathbb{R}^{n\times n}$, we define $\mathrm{He}(A)=A+A^T$. The symmetric entries in a symmetric matrix are denoted by $*$. The symbol $L_p(a,b;\mathbb{R}^n)(p=1,2,\dots)$ is the Banach space of functions x: $(a,b)\to\mathbb{R}^n$ with $\|x\|_{L^p}=\int_a^b|x(s)|^p e^{-x}ds<+\infty$.

2 Problem formulation and preliminaries

In this section, the considered system model is presented as well as some preliminaries results on Laguerre polynomials.

2.1 Systems description

Consider the linear continuous-time systems with an infinite distributed delay:

$$\dot{x}(t)=Ax(t)+A_1\int_0^{+\infty}K(\theta)x(t-\theta-h)d\theta, \quad (1)$$

where $x(t)\in\mathbb{R}^n$ is the instantaneous state vector, A, $A_1\in\mathbb{R}^{n\times n}$ are constant system matrices, and $h\geq 0$ represents a fixed time gap. It is assumed that the kernel function $K(\theta)$ has a form of

$$K(\theta)=\tilde{\Psi}(\theta)e^{-\frac{\theta}{T}},$$

where $T>0$ is a scale parameter, the function $\tilde{\Psi}(\theta)\in L_2(0,+\infty;\mathbb{R})$ and $\int_0^{+\infty}K(\theta)d\theta<+\infty$. The matrices A and $A+A_1\int_0^{+\infty}K(\theta)d\theta$ are not required to be Hurwitz. The initial condition is given by $\phi\in C^1(-\infty,0]$, where $C^1(-\infty,0]$ denotes the space of continuously differentiable functions $\phi:(-\infty,0]\to\mathbb{R}^n$ with the norm $\|\phi\|_{C^1}=\|\phi\|_C+\|\dot{\phi}\|_C<+\infty$, $\|\phi\|_C=\sup_{s\in(-\infty,0]}|\phi(s)|<+\infty$.

Let $\frac{\theta}{T}=u$. Then the system (1) can be transformed into

$$\dot{x}(t)=Ax(t)+A_1\int_0^{+\infty}\Psi(u)e^{-u}x(t-Tu-h)du, \quad (2)$$

where

$$\Psi(u)=T\tilde{\Psi}(Tu). \quad (3)$$

2.2 Laguerre polynomials and kernel approximation

In this section, the Laguerre polynomials and some relevant properties are presented. The Laguerre polynomials considered over the interval $[0,+\infty)$ are defined as follows:

$$\forall k\in\mathbb{Z}^+,\ L_k(u)=\sum_{l=0}^{k}\begin{pmatrix}k\\ l\end{pmatrix}\frac{(-1)^l}{l!}u^l. \quad (4)$$

For any functions $f, g \in L_2(0, +\infty; \mathbb{R})$, consider the inner product defined by

$$< f, g > = \int_0^{+\infty} f(\theta) g(\theta) e^{-\theta} d\theta. \qquad (5)$$

The following properties of the Laguerre polynomials presented in (4) will be helpful in deriving the main result of the paper.

Property 1 *The Laguerre polynomials form an orthonormal sequence with respect to the inner product (5) and satisfy, for all $(k, l) \in \mathbb{Z}^+ \times \mathbb{Z}^+$,*

$$\int_0^{+\infty} L_k(u) L_l(u) e^{-u} du = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \qquad (6)$$

Property 2 *For all $k \in \mathbb{Z}^+$, the Laguerre polynomials possess the following boundary conditions*

$$\begin{aligned} & L_k(0) = 1, \\ & \lim_{\theta \to 0} L_k(\theta) e^{-\theta} = 1, \\ & \lim_{\theta \to +\infty} L_k(\theta) e^{-\theta} = 0. \end{aligned} \qquad (7)$$

Property 3 *The Laguerre polynomials satisfy the following differentiation rule*

$$\dot{L}_k(u) = \begin{cases} 0, & \text{if} \quad k = 0, \\ -\sum_{i=0}^{k-1} L_i(u), & \text{if} \quad k \geq 1, \ k \in \mathbb{N}. \end{cases} \qquad (8)$$

Consider the polynomial approximation of the kernel function $\Psi(u)$. For a given integer N, $\Psi(u)$ can be rewritten as

$$\begin{aligned} \Psi(u) &= \sum_{k=0}^{N} \frac{< \Psi(\theta), \ L_k(\theta) >}{< L_k(\theta), \ L_k(\theta) >} L_k(u) + r_N(\Psi, u) \\ &= \sum_{k=0}^{N} \Psi_k L_k(u) + r_N(\Psi, u), \end{aligned} \qquad (9)$$

where

$$\begin{aligned} & \Psi_k = < \Psi(u), \ L_k(u) > = \int_0^{+\infty} \Psi(u) L_k(u) e^{-u} du, \\ & r_N(\Psi, u) = \Psi(u) - \sum_{k=0}^{N} \Psi_k L_k(u). \end{aligned} \qquad (10)$$

The term $\sum_{k=0}^{N} \Psi_k L_k(u)$ represents the projection of $\Psi(u)$ to the polynomial set $\{L_k, k = 0, \ldots, N\}$ with respect to the inner product (5). The function $r_N(\Psi, u)$ stands for the remainder of the approximation and is orthogonal to the first $N + 1$ Laguerre polynomials since from Property 1 we have for any $k = 0, \ldots, N$,

$$\begin{aligned} & \int_0^{+\infty} r_N(\Psi, \theta) L_k(\theta) e^{-\theta} d\theta = \int_0^{+\infty} \Psi(\theta) L_k(\theta) e^{-\theta} d\theta \\ & \quad - \int_0^{+\infty} \sum_{i=0}^{N} \Psi_i L_i(\theta) L_k(\theta) e^{-\theta} d\theta \\ & = \int_0^{+\infty} \Psi(\theta) L_k(\theta) e^{-\theta} d\theta - \int_0^{+\infty} \Psi_k L_k^2(\theta) e^{-\theta} d\theta \\ & = \Psi_k - \Psi_k = 0. \end{aligned} \qquad (11)$$

Let β_N denote the square of norm of r_N associated with the inner product (5), i.e.,

$$\beta_N(\Psi) = < r_N(\Psi, \theta), \ r_N(\Psi, \theta) > = \int_0^{+\infty} r_N^2(\Psi, \theta) e^{-\theta} d\theta. \qquad (12)$$

The properties of β_N are provided in the following lemma.

Lemma 1 *For $\Psi(u)$ given by (3), the following inequalities hold*

$$\beta_N(\Psi) = \int_0^{+\infty} \Psi^2(\theta) e^{-\theta} d\theta - \sum_{k=0}^{N} \Psi_k^2 \geq 0, \qquad (13)$$

$$\beta_{N+1}(\Psi) - \beta_N(\Psi) = -\Psi_{N+1}^2 \leq 0. \qquad (14)$$

Proof 1 *The proof of (13) follows the definition of $\beta_N(\Psi)$ and Property 1 of the Laguerre polynomials. The inequality (14) results directly from (13).*

2.3 Modeling of systems with infinite distributed delays

Using the Laguerre polynomials approximation (9) of the kernel function $\Psi(u)$, system (2) is equivalent to

$$\dot{x}(t) = Ax(t) + A_1 \sum_{k=0}^{N} \Psi_k \Omega_k(x_t) + \beta_N(\Psi) A_1 \Xi_N(\Psi, x_t), \qquad (15)$$

where

$$\begin{aligned} & \Omega_k(x_t) = \int_0^{+\infty} L_k(\theta) x(t - T\theta - h) e^{-\theta} d\theta, \\ & \Xi_N(\Psi, x_t) = \begin{cases} 0, \quad \text{if } \beta_N(\Psi) = 0, \\ \frac{1}{\beta_N(\Psi)} \int_0^{+\infty} r_N(\Psi, \theta) x(t - T\theta - h) e^{-\theta} d\theta, \\ \text{if } \beta_N(\Psi) > 0. \end{cases} \end{aligned} \qquad (16)$$

Remark 1 *In [10] and [11], to assess stability of systems with particular gamma-distributed delays, whose model is given in (1) with $K(\theta)$ having a form of*

$$K(\theta) = \frac{\theta^{M-1} e^{-\frac{\theta}{T}}}{T^M (M-1)!}, \tag{17}$$

where $M \geq 2$, $M \in \mathbb{N}$, is a scaling parameter of the distribution and $T > 0$ is a scale parameter, one needs to transform the original system (1) with (17) into the following augmented one:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 y(t), \\ \dot{y}(t) &= -\tfrac{1}{T} y(t) + \rho(t), \end{aligned} \tag{18}$$

where

$$\begin{aligned} y(t) &= \int_0^{+\infty} K(\theta) x(t-\theta-h) d\theta, \\ \rho(t) &= \int_0^{+\infty} Q(\theta) x(t-\theta-h) d\theta, \\ Q(\theta) &= \frac{\theta^{M-2} e^{-\frac{\theta}{T}}}{T^M (M-2)!}. \end{aligned}$$

This may lead to conservatism (smaller stability region in the (T,h) plane) because the stability of the augmented system (18) implies the stability of the original system (1) with (17), but not vice versa [10].

In the present paper, the original system (1) is not necessary to be transformed into an augmented one. Instead it is represented as a system with additional signals Ω_k, $k = 0, \ldots, N$, and Ξ_N that can be captured by the newly introduced Bessel-Laguerre integral inequality proposed in the next section.

3 Bessel-Laguerre integral inequality

In this section, taking advantages of the properties of Laguerre polynomials and applying Bessel inequality [18], we provide an integral inequality, which includes the additional signals of system (15) and then allows us to develop efficient stability criterion.

Lemma 2 *(Bessel-Laguerre integral inequality) For a given $n \times n$ matrix $R \succ 0$, and $x(\theta) \in L_2(0, +\infty; \mathbb{R}^n)$, the following inequality*

$$\begin{aligned} \int_0^{+\infty} x^T(\theta) R x(\theta) e^{-\theta} d\theta \geq \sum_{k=0}^{N} \Omega_k^T(x) R \Omega_k(x) \\ + \rho(\beta_N(\Psi)) \Xi_N^T(\Psi, x) R \Xi_N(\Psi, x) \end{aligned} \tag{19}$$

holds, where the notations $\Omega_k(x)$ and $\Xi_N(\Psi, x)$ are defined in (16) and

$$\rho(\beta_N(\Psi)) = \begin{cases} \beta_N(\Psi), & \text{if } \beta_N(\Psi) > 0, \\ 1, & \text{if } \beta_N(\Psi) = 0. \end{cases} \tag{20}$$

Proof 2 *For $x \in L_2(0, +\infty; \mathbb{R}^n)$, $\Psi \in L_2(0, +\infty; \mathbb{R})$ and Laguerre polynomials L_k, $k = 0, \ldots, N$, define a function $f_N(\theta)$ for all $\theta \in [0, +\infty)$ by*

$$f_N(\theta) = x(\theta) - \underbrace{\sum_{k=0}^{N} \Omega_k(x) L_k(\theta)}_{x_N(\theta)} - \underbrace{r_N(\Psi, \theta) \Xi_N(\Psi, x)}_{\nu_N(\theta)}. \tag{21}$$

The above definition indicates that function $f_N(\theta)$ belongs to $L_2(0, +\infty; \mathbb{R}^n)$, and thus, $\int_0^{+\infty} f_N^T(\theta) R f_N(\theta) e^{-\theta} d\theta$ exists. In the next developments and for the sake of simplicity, we will denote by $\mathcal{I}(f)$ the integral $\int_0^{+\infty} f^T(\theta) R f(\theta) e^{-\theta} d\theta$, for any appropriate function f. Since matrix R is positive definite it is clear that $\mathcal{I}(f_N)$ is non-negative. Expanding the expression of f_N yields

$$\begin{aligned} 0 \leq \mathcal{I}(f_N) &= \mathcal{I}(x) + \mathcal{I}(x_N) + \mathcal{I}(\nu_N) \\ &-2 \int_0^{+\infty} x^T(\theta) R x_N(\theta) e^{-\theta} d\theta - 2 \int_0^{+\infty} x^T(\theta) R \nu_N(\theta) e^{-\theta} d\theta \\ &+2 \int_0^{+\infty} x_N^T(\theta) R \nu_N(\theta) e^{-\theta} d\theta. \end{aligned} \tag{22}$$

Our objective is to simplify this expression. Consider the case that $\beta_N(\Psi)$, and consequently, r_N are not zero. Let us first note that

$$\mathcal{I}(x_N) = \sum_{k=0}^{N} \sum_{j=0}^{N} \left(\int_0^{+\infty} L_k(\theta) L_j(\theta) e^{-\theta} d\theta \right) \Omega_k^T(x) R \Omega_j(x).$$

Using the orthonormal property (6), the previous expression simplifies to $\mathcal{I}(x_N) = \sum_{k=0}^{N} \Omega_k^T(x) R \Omega_k(x)$. Similarly, we note that

$$\begin{aligned} \mathcal{I}(\nu_N) &= \left(\int_0^{+\infty} r_N^2(\Psi, \theta) e^{-\theta} d\theta \right) \Xi_N^T(\Psi, x) R \Xi_N(\Psi, x) \\ &= \beta_N(\Psi) \Xi_N^T(\Psi, x) R \Xi_N(\Psi, x). \end{aligned}$$

Then, the fourth term of the right-hand-side of (22) can *be rewritten by expanding the expression of* x_N

$$\begin{aligned}
&-2\int_0^{+\infty} x^T(\theta)Rx_N(\theta)e^{-\theta}d\theta \\
&= -2\sum_{k=0}^{N}\left(\int_0^{+\infty} L_k(\theta)e^{-\theta}x(\theta)d\theta\right)^T R\Omega_k(x) \\
&= -2\sum_{k=0}^{N}\Omega_k^T(x)R\Omega_k(x) \\
&= -2\mathcal{I}(x_N).
\end{aligned}$$

Similarly, we get that

$$\begin{aligned}
&-2\int_0^{+\infty} x^T(\theta)R\nu_N(\theta)e^{-\theta}d\theta \\
&= -2\left(\int_0^{+\infty} r_N(\Psi,\theta)e^{-\theta}x(\theta)d\theta\right)^T R\Xi_N(\Psi,x) \\
&= -2\beta_N(\Psi)\Xi_N^T(\Psi,x)R\Xi_N(\Psi,x) \\
&= -2\mathcal{I}(\nu_N).
\end{aligned}$$

It remains to express the last term of the right-hand-side of (22)*. This yields*

$$\begin{aligned}
&2\int_0^{+\infty} x_N^T(\theta)R\nu_N(\theta)e^{-\theta}d\theta \\
&= 2\sum_{k=0}^{N}\left(\int_0^{+\infty} r_N(\Psi,\theta)L_k(\theta)e^{-\theta}d\theta\right)\Omega_k^T(x)R\Xi_N(\Psi,x).
\end{aligned}$$

In light of the construction of r_N *in* (11)*, we straightforwardly obtain that this term is zero. Hence, re-injecting the previous expressions into* (22) *yields*

$$0 \le \mathcal{I}(x) - \mathcal{I}(x_N) - \mathcal{I}(\nu_N),$$

which concludes the proof for the case $\beta_N(\Psi) > 0$*. Furthermore, in the case where* $\beta_N(\Psi)$*, and consequently,* r_N *are zero, from* (22) *it follows that*

$$\int_0^{+\infty} x^T(\theta)Rx(\theta)e^{-\theta}d\theta \ge \sum_{k=0}^{N}\Omega_k^T(x)R\Omega_k(x),$$

which yields (19) due to $\Xi_N(\Psi,x)$ *given by (16) is zero when* $\beta_N(\Psi) = 0$*.*

Remark 2 *The well-known Jensen inequality was generalized in [10] from finite intervals of integration to infinite ones: given an* $n\times n$ *matrix* $R \succ 0$ *and a scalar function* $K : [0,+\infty) \to \mathbb{R}^+$ *such that the integrations concerned are well defined, the following inequality*

$$\begin{aligned}
&\int_0^{+\infty} K(s)x^T(s)Rx(s)ds \\
&\ge K_0^{-1}\int_0^{+\infty} K(s)x^T(s)dsR\int_0^{+\infty} K(s)x(s)ds
\end{aligned} \tag{23}$$

holds, where $K_0 = \int_0^{+\infty} K(s)ds$*. Recently, the inequality (23) was extended in [11] to a more general form:*

$$\begin{aligned}
&\int_0^{+\infty} K(s)x^T(s)Rx(s)ds \\
&\quad\ge K_0^{-1}\int_0^{+\infty} K(s)x^T(s)dsR\int_0^{+\infty} K(s)x(s)ds \\
&\quad\quad+\left(K_2 - \tfrac{K_1^2}{K_0}\right)^{-1}\tilde{\Omega}^T R\tilde{\Omega},
\end{aligned} \tag{24}$$

where $K_1 = \int_0^{+\infty} sK(s)ds$*,* $K_2 = \int_0^{+\infty} s^2K(s)ds$ *and* $\tilde{\Omega} = \frac{K_1}{K_0}\int_0^{+\infty} K(s)x(s)ds - \int_0^{+\infty} sK(s)x(s)ds$*. Note that the Bessel-Laguerre inequality (19) with* $\Xi_N(\Psi,x) = 0$ *for* $N = 0$ *and* $N = 1$ *allows retrieving inequalities (23) and (24) with* $K(s) = e^{-s}$*, respectively. This remark demonstrates the generality of the Bessel-Laguerre inequality.*

4 Stability analysis

In this section, the new integral inequality (19) in Lemma 2 will be employed for the stability analysis of system (1) with infinite distributed delays. For the sake of simplicity, we will use in this section the following notations:

$$\begin{aligned}
G_N &= \begin{bmatrix} I_n & 0_{n,n} & 0_{n,n(N+1)} \\ 0_{n(N+1),n} & 0_{n(N+1),n} & I_{n(N+1)} \end{bmatrix}, \\
F_N &= \begin{bmatrix} A & 0_{n,n} & A_1\Psi_0 & A_1\Psi_1 & \cdots & A_1\Psi_N \end{bmatrix}, \\
W_N &= \begin{bmatrix} F_N^T & \Upsilon_N^T(0) & \Upsilon_N^T(1) & \cdots & \Upsilon_N^T(N) \end{bmatrix}^T, \\
\Upsilon_N(k) &= \begin{bmatrix} 0_{n,n} & \frac{1}{T}I_n & r_{Nk}^0 I_n & r_{Nk}^1 I_n & \cdots & r_{Nk}^N I_n \end{bmatrix}, \\
r_{Nk}^i &= \begin{cases} -\frac{1}{T}, & if\ i \le k, \\ 0, & if\ i \ge k+1. \end{cases}
\end{aligned} \tag{25}$$

We propose to adopt the following Lyapunov functional:

$$\begin{aligned}
V_N(x_t) = \zeta_N^T(x_t)P_N\zeta_N(x_t) + h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)S\dot{x}(s)dsd\theta \\
+\int_0^{+\infty}\int_{t-T\theta-h}^{t} e^{-\theta}x^T(s)Rx(s)dsd\theta,
\end{aligned} \tag{26}$$

where $P_N \succ 0$, $S \succ 0$, $R \succ 0$ and the augmented vector $\zeta_N(x_t)$ is given by

$$\zeta_N(x_t) = \begin{bmatrix} x_t(0) \\ \int_0^{+\infty} L_0(\theta)x(t-T\theta-h)e^{-\theta}d\theta \\ \vdots \\ \int_0^{+\infty} L_N(\theta)x(t-T\theta-h)e^{-\theta}d\theta \end{bmatrix} = \begin{bmatrix} x_t(0) \\ \Omega_0(x_t) \\ \vdots \\ \Omega_N(x_t) \end{bmatrix}. \quad (27)$$

We now state the main result of this paper:

Theorem 1 *Given $h > 0$, assume that there exist $(n(N+2)) \times (n(N+2))$ positive definite matrix P_N and $n \times n$ positive definite matrices S, R such that the following matrix inequality*

$$\begin{bmatrix} \Phi_{N0} & \beta_N(\Psi)G_N^T P_N J_N A_1 & h^2 F_N^T S \\ * & -\rho(\beta_N(\Psi))R & h^2\beta_N(\Psi)A_1^T S \\ * & * & -h^2 S \end{bmatrix} \prec 0 \quad (28)$$

is satisfied, where

$$\begin{aligned} \Phi_{N0} &= \mathrm{He}(G_N^T P_N W_N) + \Pi_N - E_N^T S E_N - R_N, \\ \Pi_N &= \mathrm{diag}(R, 0_{n(N+2),n(N+2)}), \\ R_N &= \mathrm{diag}(0_{2n,2n}, \underbrace{R, \dots, R}_{N+1}), \\ E_N &= \begin{bmatrix} I_n & -I_n & 0_{n,n(N+1)} \end{bmatrix}, \\ J_N &= \begin{bmatrix} I_n & 0_{n,n(N+1)} \end{bmatrix}^T \end{aligned} \quad (29)$$

with the other notations given in (25). Then system (1) is asymptotically stable for the time gap h.

Proof 3 *Differentiating $V_N(x_t)$ along (1) and applying Jensen inequality, we have*

$$\begin{aligned} \dot{V}_N(x_t) \le\ & 2\zeta_N^T(x_t)P_N\dot{\zeta}_N(x_t) \\ & + x^T(t)Rx(t) + h^2\dot{x}^T(t)S\dot{x}(t) \\ & - [x(t)-x(t-h)]^T S[x(t)-x(t-h)] \\ & - \int_0^{+\infty} x^T(t-T\theta-h)Rx(t-T\theta-h)e^{-\theta}d\theta. \end{aligned} \quad (30)$$

The objective of the next developments consists in finding an upper bound of $\dot{V}_N(x_t)$ using $\Xi_N(\Psi, x_t)$ and the augmented vector $\xi_N(x_t)$ given by

$$\xi_N(x_t) = \mathrm{col}\{x(t), x(t-h), \Omega_0(x_t), \cdots, \Omega_N(x_t)\}. \quad (31)$$

Consider the first term of the right hand side of (30). It is easy to see that

$$\dot{\zeta}_N(x_t) = \begin{bmatrix} \dot{x}_t(0) \\ \dot{\Omega}_0(x_t) \\ \vdots \\ \dot{\Omega}_N(x_t) \end{bmatrix} = \begin{bmatrix} \dot{x}_t(0) \\ \Omega_0(\dot{x}_t) \\ \vdots \\ \Omega_N(\dot{x}_t) \end{bmatrix}.$$

From (15) and the definition of F_N, it follows that

$$\dot{x}_t(0) = F_N\xi_N(x_t) + \beta_N(\Psi)A_1\Xi_N(\Psi, x_t).$$

Then, an integration by parts ensures that for any $k = \{0, \dots, N\}$,

$$\begin{aligned} \Omega_k(\dot{x}_t) =\ & -\tfrac{1}{T}[\lim_{\theta\to+\infty}(L_k(\theta)x(t-T\theta-h)e^{-\theta})] \\ & + \tfrac{1}{T}[\lim_{\theta\to 0}(L_k(\theta)x(t-T\theta-h)e^{-\theta})] \\ & + \frac{1}{T}\int_0^{+\infty} \dot{L}_k(\theta)x(t-T\theta-h)e^{-\theta}d\theta \\ & - \frac{1}{T}\int_0^{+\infty} L_k(\theta)x(t-T\theta-h)e^{-\theta}d\theta. \end{aligned}$$

Thanks to Properties 2 and 3 of the Laguerre polynomials, the following expression is derived

$$\begin{aligned} \Omega_k(\dot{x}_t) &= \frac{1}{T}x(t-h) - \frac{1}{T}\sum_{i=0}^{k}\int_0^{+\infty} L_i(\theta)x(t-T\theta-h)e^{-\theta}d\theta \\ &= \frac{1}{T}x(t-h) - \frac{1}{T}\sum_{i=0}^{k}\Omega_i(x_t) \\ &= \Upsilon_N(k)\xi_N(x_t). \end{aligned}$$

Collecting all the components of $\dot{\zeta}_N(x_t)$, we have

$$\dot{\zeta}_N(x_t) = W_N\xi_N(x_t) + \beta_N(\Psi)J_N A_1\Xi_N(\Psi, x_t),$$

where the matrices W_N and J_N are given in (25) and (29), respectively. Next, applying Lemma 2 to the last term of (30), we obtain the following inequality

$$\begin{aligned} & -\int_0^{+\infty} x^T(t-T\theta-h)Rx(t-T\theta-h)e^{-\theta}d\theta \\ & \le -\sum_{k=0}^{N}\Omega_k^T(x_t)R\Omega_k(x_t) - \rho(\beta_N(\Psi))\Xi_N^T(\Psi, x_t)R\Xi_N(\Psi, x_t) \\ & = -\xi_N^T(x_t)R_N\xi_N(x_t) - \rho(\beta_N(\Psi))\Xi_N^T(\Psi, x_t)R\Xi_N(\Psi, x_t). \end{aligned}$$

Since $\zeta_N(x_t){=}G_N\xi_N(x_t)$ *and* $x^T(t)Rx(t){=}\xi_N^T(x_t)\Pi_N\xi_N(x_t)$ *and* $(x(t){-}x(t{-}h))^TS(x(t){-}x(t{-}h)){=}\xi_N^T(x_t)E_N^TSE_N\xi_N(x_t)$ *it follows that*

$$\begin{aligned}\dot{V}_N(x_t) \leq & 2\xi_N^T(x_t)G_N^TP_N[W_N\xi_N(x_t){+}\beta_N(\Psi)J_NA_1\Xi_N(\Psi,x_t)] \\ & +h^2\dot{x}^T(t)S\dot{x}(t){+}\xi_N^T(x_t)(\Pi_N{-}R_N{-}E_N^TSE_N)\xi_N(x_t) \\ & -\rho(\beta_N(\Psi))\Xi_N^T(\Psi,x_t)R\Xi_N(\Psi,x_t) \\ = & \eta_N^T(x_t)\begin{bmatrix}\Phi_{N0} & \beta_N(\Psi)G_N^TP_NJ_NA_1 \\ * & -\rho(\beta_N(\Psi))R\end{bmatrix}\eta_N(x_t) \\ & +h^2[F_N\xi_N(x_t)+\beta_N(\Psi)A_1\Xi_N(\Psi,x_t)]^TS \\ & \times[F_N\xi_N(x_t)+\beta_N(\Psi)A_1\Xi_N(\Psi,x_t)],\end{aligned} \tag{32}$$

where $\eta_N(x_t) = \mathrm{col}\{\xi_N(x_t),\ \Xi_N(\Psi,x_t)\}$ *and other notations are given by (25) and (29). Hence, by Schur complements, (28) guarantees* $\dot{V}_N(x_t) \leq -\varepsilon|x(t)|^2$ *for some* $\varepsilon > 0$, *which implies the asymptotic stability of system (1) with infinite distributed delays.*

Remark 3 *In some situations, the stability analysis of time-delay systems introduces additional dynamics that may introduce conservatism. Let us point out that the Bessel-Laguerre and more generally Bessel methods do not introduce any additional dynamics to the original systems. It only corresponds to manipulations, that allows providing more accurate analysis and less conservative results than existing ones.*

Remark 4 *The stability condition of Theorem 1 is parameterized by the degree* N *of the Laguerre polynomial , which indicates the precision of the polynomial approximation. Note that the order of LMIs and the number of decision variables increase with* N. *As shown in the examples below, the improvements may be achieved by increasing* N, *but at the price of additional decision variables, showing a classical tradeoff between the reduction of the conservatism and the numerical complexity.*

5 Illustrative examples

Two numerical examples from the literature will illustrate the efficiency of the proposed conditions.

5.1 Application to gamma-distributed delays

The previous method can be easily applied to systems with gamma kernel functions, whose model is given by (1) with (17). Let $\frac{\theta}{T} = u$, then system (1) with (17) can be transformed into system (2) with $\Psi(u)$ having a form of

$$\Psi(u) = \frac{u^{M-1}}{(M-1)!}. \tag{33}$$

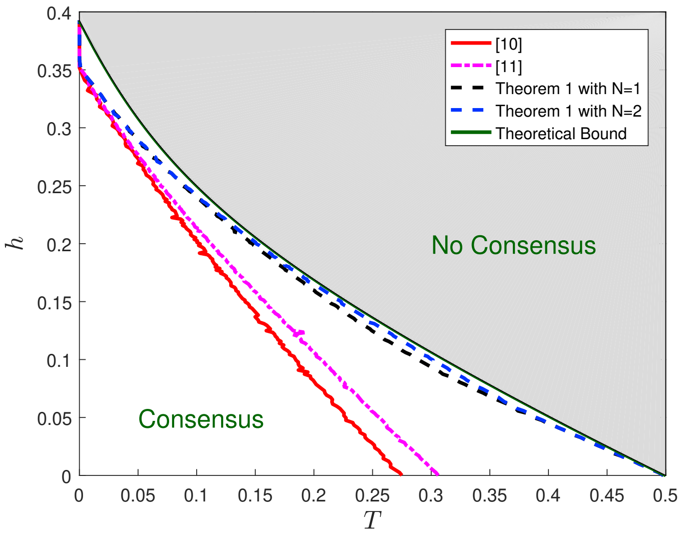

Fig. 1. Example 1: tradeoff curves between maximal allowable T and h by different methods

Consider system (1) with (17). Following [6] and [10], we give an example of two cars on a ring, where

$$A = 0 \ \text{ and } \ A_1 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

Here neither A nor $A + A_1$ is Hurwitz. The objective is to find the stability region in the (T, h) plane that preserves the asymptotic stability. Taking the scaling parameter of distribution $M = 2$ we have $\Psi(u) = u$ and then $\int_0^{+\infty} \Psi^2(u)e^{-u}du = 2$. Further, simple computation shows

$$\begin{aligned}\Psi_0 &= \int_0^{+\infty} \Psi(u)L_0(u)e^{-u}du = 1, \\ \Psi_1 &= \int_0^{+\infty} \Psi(u)L_1(u)e^{-u}du = -1, \\ \Psi_2 &= \int_0^{+\infty} \Psi(u)L_2(u)e^{-u}du = 0.\end{aligned}$$

From (13) and (14), it follows readily that $\beta_i(\Psi) = 0$, and therefore, $\rho(\beta_i(\Psi)) = 1$, $i = 1, 2$.

For different values of h given in Table 1, by applying Proposition 3 of [10] , Proposition 1 of [11], and using Theorem 1, we obtain the maximum allowable values of T that guarantee the stability. Figure 1 illustrates tradeoff curves between maximal allowable T and h by applying the above Lyapunov-based methods and by the frequency domain technique proposed in [6] for the theoretical bounds. One can see that the conditions of Theorem 1 with $N = 1$ essentially achieve better results than those of [10] and [11]. As expected in Remark 4, with additional decision variables, Theorem 1 with $N = 2$

Table 1
Example 1: maximum allowable value of T for different h.

$[\max T]\backslash\ h$	10^{-5}	0.01	0.03	0.05	0.08	0.1	0.15	0.2	0.35	Decision variables
[10]	0.274	0.265	0.248	0.229	0.199	0.179	0.140	0.102	-	22
[11]	0.305	0.296	0.276	0.256	0.226	0.205	0.158	0.110	0.002	16
Th 1($N=1$)	0.495	0.467	0.432	0.390	0.325	0.289	0.213	0.147	0.002	27
Th 1($N=2$)	0.499	0.477	0.433	0.390	0.333	0.301	0.223	0.148	0.002	42
Re 5($N=2$)	0.499	0.477	0.433	0.390	0.333	0.301	0.223	0.151	0.002	45

slightly improves the results, which are very close to analytical ones except for smaller values of T.

Remark 5 *The method of [10] and [11] for stability analysis of system with gamma-distributed delays requires one more triple integral term in Lyapunov functional, which is not essential in our analysis. However, following [10] and [11], we can insert one additional term*

$$V_U(x_t) = \int_0^{+\infty} \int_0^{T\theta+h} \int_{t-\lambda}^{t} e^{-\theta}\dot{x}^T(s)U\dot{x}(s)dsd\lambda d\theta, U \succ 0,$$

into Lyapunov functional (26). Differentiation of V_U and applying the double integral inequality (see e.g., [10] and [11]) yield

$$\dot{V}_U(x_t) \leq (T+h)\dot{x}^T(t)U\dot{x}(t) - \frac{1}{h+1}\xi_N^T(x_t)Y_N^T U Y_N \xi_N(x_t),$$

where $\xi_N(x_t)$ is given by (31) and $Y_N = [I_n\ 0_{n,n}\ -I_n\ 0_{n,nN}]$. The stability result derived by $V_N(x_t)+V_U(x_t)$ almost coincides with the one by Theorem 1 (see the Example 1).

Remark 6 *The stability analysis in [11] for linear systems with gamma-distributed delays was based on a generalized integral inequality and its double integral extension. Note that the method of [11] is efficient only for systems with gamma kernel functions ($M \geq 2$) and not suitable for systems with more general kernels. It is worth mentioning that compared with [11], the proposed method in this paper can not only deal with systems with more general kernels but also lead to less conservative results for systems with gamma-distributed delays (larger stability region in the (T, h) plane).*

5.2 Example 2

Consider system (1) from [10], where

$$A = 0 \text{ and } A_1 = -41.8.$$

The kernel function $\tilde{\Psi}(\theta) \in L_2(0, +\infty; \mathbb{R})$ is given by $\tilde{\Psi}(\theta) = \frac{3+20\theta+700\theta^2}{2-20\theta+800\theta^2}$ and $T = \frac{1}{10}$. As noted in Remark 6, the method of [11] is not applicable to this example since the kernel function is not a gamma distribution. Applying Proposition 1 of [10], the maximum value of h is found to be 0.128. The conditions of Theorem 1 with $N = 0, 1, 2$ and 3 improve the results and yield the maximum values of h to be 0.146, 0.167, 0.174 and 0.175, respectively. It is demonstrated that the results become better with the increase of degree of the polynomial N.

6 Conclusions

In this paper, we have studied the stability of linear systems with infinite distributed delays and with general kernels. The analysis is based on a Laguerre polynomials approximation of kernels and the newly introduced Bessel-Laguerre integral inequality. By using a Lyapunov method, a set of sufficient conditions has been derived that is dependent on the degree N of the polynomial. The suggested framework can easily include the case of gamma kernel functions. Numerical examples illustrate the potential improvements achieved by proposed conditions with increasing N, but at the price of additional decision variables. In particular, for the traffic flow models on the ring, the proposed results are shown to be very close to analytical ones. Future research includes other applications of this novel inequality.

References

[1] W. Chen and W. Zheng, "Delay-dependent robust stabilization for uncertain neutral systems with distributed delays," *Automatica*, vol. 43, no. 1, pp. 95–104, 2007.

[2] Z. Feng and J. Lam, "Integral partitioning approach to robust stabilization for uncertain distributed time-delay systems," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 6, pp. 676–689, 2012.

[3] E. Fridman, *Introduction to Time-Delay Systems*. Birkhäuser, 2014.

[4] S. Xu, J. Lam, T. Chen, and Y. Zou, "A delay-dependent approach to robust H_∞ filtering for uncertain distributed delay systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 10, pp. 3764–3772, 2005.

[5] W. Michiels, C. I. Morarescu, and S. I. Niculescu, "Consensus problems with distributed delays, with application to traffic flow models," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 77–101, 2009.

[6] C. Morarescu, S. Niculescu, and K. Gu, "Stability crossing curves of shifted gamma-distributed delay systems," *SIAM*

Journal on Applied Dynamical Systems, vol. 6, no. 2, pp. 475–493, 2007.

[7] G. Wei, G. Feng, and Z. Wang, "Robust H_∞ control for discrete-time fuzzy systems with infinite-distributed delays," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 1, pp. 224–232, 2009.

[8] O. Roesch, H. Roth, and S. Niculescu, "Remote control of mechatronic systems over communication networks," in *Proceedings of the IEEE International Conference on Mechatronics & Automation*, Niagara Falls, Canada, July 2005.

[9] R. Sipahi, F. M. Atay, and S. I. Niculescu, "Stability of traffic flow behavior with distributed delays modeling the memory effects of the drivers," *SIAM Journal on Applied Mathematics*, vol. 68, no. 3, pp. 738–759, 2007.

[10] O. Solomon and E. Fridman, "New stability conditions for systems with distributed delays," *Automatica*, vol. 49, no. 11, pp. 3467–3475, 2013.

[11] K. Liu, E. Fridman, K. H. Johansson, and Y. Xia, "Generalized Jensen inequalities with application to stability analysis of systems with distributed delays over infinite time-horizons," *Automatica*, vol. 69, pp. 222–231, 2016.

[12] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems.* Birkhäuser, Boston, 2003.

[13] A. Seuret and F. Gouaisbaut, "Wirtinger-based integral inequality: application to time-delay systems," *Automatica*, vol. 49, no. 9, pp. 2860–2866, 2013.

[14] F. Gouaisbaut, Y. Ariba, and A. Seuret, "Bessel inequality for robust stability analysis of time-delay system," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, Firenze, Italy, December 2013.

[15] A. Seuret and F. Gouaisbaut, "Hierarchy of LMI conditions for the stability analysis of time-delay systems," *Systems & Control Letters*, vol. 81, pp. 1–7, 2015.

[16] A. Seuret, F. Gouaisbaut, and Y. Ariba, "Complete quadratic Lyapunov functionals for distributed delay systems," *Automatica*, vol. 62, pp. 168–176, 2015.

[17] A. Seuret and F. Gouaisbaut, "Stability of linear systems with time-varying delays using Bessel-Legendre inequalities," *IEEE Transactions on Automatic Control*, vol. 63, no. 1, pp. 225–232, 2018.

[18] W. Gautschi, *Orthogonal polynomials, computation and approximation.* Oxford Science, 2004.