Continuous quaternion based almost global attitude tracking with LMI multi-objective tuning

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Abstract

This paper considers the attitude control problem of a generic rotating 3 degrees of freedom fully actuated rigid object. The specific studied problem is the deviation control of this object around a theoretically feasible attitude trajectory. The rotation motion has an intrinsic non linear behaviour (trigonometric, 2π-periodicity) that may lead to build non linear and hybrid controllers. This paper considers the opportunity to use the quaternion framework to build a continuous non linear controller that reaches an almost global asymptotical stability.

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Contents

1 Introduction

This paper studies the attitude control problem of a rigid fully actuated object in the unit quaternion framework. The attitude control of a rigid object is a widely studied problem as it is a central issue of all moving robots: aircrafts, drones, spacecrafts, satellites, manipulators.

The attitude control presents some specific complexity linked to the topology of the rotational motion : its trigonometric behaviour and its 2π-periodicity. The paper [5] describes quite exhaustively the various mathematical frameworks existing to model the attitude of an object: Euler angles, rotation matrix, unit quaternion. As demonstrated in [4], this topological issue leads to not be able to build a global stable static or dynamic continuous linear time invariant control law for the attitude control of a rigid object.

The unit quaternion framework is chosen here because of its efficiency and compactness: 4 parameters against 9 for the rotation matrix, it does not involve direct trigonometry developments, and it suits the state space framework. The topological issue remains and appears as a singularity problem called *double coverage*: a unit quaternion q and its opposite $-q$ represents the same object attitude (cf. [15] for detailed quaternion algebra). This quaternion singularity problem may generate an *unwinding phenomenon*: the controlled object flips back all the way around whereas it was just nearby the required attitude, but it had done the travel rotating from the other side and it does again the same travel backwards.

From the general mathematical "stability of motion" problem which can be formalized with the Lyapunov theory as in [9], the motion control of a rigid object is included in the class of Lagrangian mechanical non linear systems as shown in [17]. For this class of systems, many control strategies have been studied, starting with a reference result [18] in 1988 with a dynamics reinjection input and a Proportional Derivative (PD) deviation controller architecture.

The quaternion-based attitude control problem is still nowadays widely studied. Some approaches study similar strategy as previous Lagrangian PD control architecture [12, 2]. Others build some non linear controllers that match the non linearities and singularities of the quaternion attitude kinematics with specific Lyapunov function [22, 10, 23, 24, 1]. The singularity issue is also managed with an hybrid control strategy depending on the sign of the first component of the unit quaternion as in [13]. Some others address the robustness of the controller against disturbance input or parametric uncertainties [11, 8, 14].

As developped in section 2, the novelty of this paper is to consider the attitude trajectory deviation dynamics as a single quaternion-based non linear state space system. It brings out in section 3 the opportunity to build continuous non linear controller structure. The demonstration of the closed loop system almost global asymptotic stability as defined in [16] is done with a relaxation of conditions of Input to State Stability (ISS) as demonstrated in [7]. We illustrate the system almost global stability property in section 4 with an equivalent physical assembly and a phase-plane simulation. We eventually propose in section 5 an LMI procedure as in [6] to make the synthesis of the controller according to multi-objective performances requirements, illustrated with a real time simulation in 6.

2 Attitude deviation model

2.1 Newton's law for the rotation motion

The Newton's law for the attitude dynamics applied to the center of gravity of an object of inertia $J_b \in \mathbb{R}^{3\times 3}$, constant symmetric definite positive matrix, corresponds to the following set of differential equations (cf. [15] for details on rotation motion and quaternion):

$$\begin{aligned} \frac{d(q)}{dt} &= \frac{1}{2}\begin{pmatrix} -q_V \\ S_q(q) \end{pmatrix}\omega_b \\ J_b\frac{d(\omega_b)}{dt} + S_V(\omega_b)J_b\omega_b &= C_{act_b} + C_{ext_b} \end{aligned} \tag{1}$$

With all the variables without index expressed in the earth referential, and all the variables with the index "$_b$" expressed in the rigid body referential, with:

- the instantaneous attitude represented as the unit quaternion:

$$q = \begin{pmatrix} q_o \\ q_V \end{pmatrix} = \begin{pmatrix} q_o \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} cos(\theta/2) \\ sin(\theta/2) \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \end{pmatrix} \in \mathbb{R}^4$$

  which corresponds to a rotation of the object of an angle θ around the axis defined by the unit vector $n = (n_x \; n_y \; n_z)^\top \in \mathbb{R}^3$. q_o is generally called the *scalar part* of the quaternion and q_V the *vector part*. The *double coverage* issue appears in this definition as q and $-q$ represents the same attitude. A complete non ambiguous attitude position with these terms can be defined by $sign(q_o)q_V$.

- $S_q(q)$ is the skew symmetric matrix of the quaternion q defined as follows:

$$S_q(q) = \begin{pmatrix} q_o & -q_3 & q_2 \\ q_3 & q_o & -q_1 \\ -q_2 & q_1 & q_o \end{pmatrix} = q_o I_3 + S_V(q_V)$$

  The skew symmetric matrix S_V is equivalent to the vectorial product of two vectors V_1 and V_2 of $\mathbb{R}^3$ such that : $S_V(V_1)V_2 = V_1 \wedge V_2$. The matrix inverse of the quaternion skew symmetric matrix is equal to (not defined at $q_o = 0$):

$$S_q(q)^{-1} = \frac{1}{q_o} I_3 - \frac{1}{1+q_o^2} S_V(q_V) + \frac{1}{q_o(1+q_o^2)} S_V(q_V)^2$$

- $\omega_b \in \mathbb{R}^3$ is the rotation speed vector,

- C_{act_b}, $C_{ext_b} \in \mathbb{R}^3$ are respectively the torques applied by the actuators and the external environment (air drag, objects or walls in contact with the object).

The second equation is expressed in the rotating rigid body referential so that the inertia appears as the constant J_b. In this referential, the inputs are also directly equal to actuators actions (actuators attached to the rotating object). We assume to have a fully actuated object, actuators acting independently on the three components of the control input torque C_{act_b}.

2.2 State space attitude deviation model

Assumption 1. *The optimal command problem (cf. [3]) is considered solved for the theoretical system* (1) *for a thoretical inertia* J_b^**, without any external disturbances* ($C_{ext_b} = 0$)*, giving a theoretically feasible trajectory* (q^*, ω_b^*) *with its optimal input* $C_{act_b}^*$ *solution of* (1)*:*

$$\begin{aligned} \frac{d(q^*)}{dt} &= \frac{1}{2}\begin{pmatrix} -q_V^* \\ S_q(q^*) \end{pmatrix}\omega_b^* \\ J_b^*\frac{d(\omega_b^*)}{dt} + S_V(\omega_b^*)J_b^*\omega_b^* &= C_{act_b}^* \end{aligned} \tag{2}$$

All the theoretical values are noted with a * *exponent.*

Proposition 1. *The tracking deviation of* (1) *with respect to* (2) *has the following almost linear representation:*

$$\begin{aligned} H : \dot{x} &= A(x)x + B_w w + B_u u \\ z_q &= C_q x \\ z_\omega &= C_\omega x \\ y &= x \end{aligned} \tag{3}$$

With the state $x = \begin{pmatrix} q_V^{\varepsilon\top} & \omega_b^{\varepsilon^*\top} \end{pmatrix}^\top \in \mathbb{R}^6$ *and the state space matrices:*

$$A(x) = \begin{pmatrix} 0 & \frac{1}{2}S_q(q^\varepsilon) \\ 0 & 0 \end{pmatrix}, \; B_u = B_w = \begin{pmatrix} 0 \\ I_3 \end{pmatrix}$$

$$C_q = \begin{pmatrix} I_3 & 0 \end{pmatrix}, \; C_\omega = \begin{pmatrix} 0 & I_3 \end{pmatrix}$$

The scalar part of the quaternion q_o^ε *inside the matrix* $S_q(q^\varepsilon) = q_o^\varepsilon I_3 + S_V(q_V^\varepsilon)$ *in* $A(x)$ *is the non linear function of the state solution of:*

$$\frac{dq_o^\varepsilon}{dt} = -\frac{1}{2}q_V^{\varepsilon\top}\omega_b^{\varepsilon^*} = -\frac{1}{4}x^\top\begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}x \tag{4}$$

It is not considered as a state of the system. It also respects the unit norm constraint: $q_o^{\varepsilon 2} + ||q_V^\varepsilon||^2 = 1$.

Proof: The attitude deviation system H is obtained performing the non linear difference between (1) and (2) giving the following variables definitions:

- the state $x = \begin{pmatrix} q_V^{\varepsilon\top} & \omega_b^{\varepsilon^*\top} \end{pmatrix}^\top \in \mathbb{R}^6$ defined by:
  - q_V^ε the vector part of the attitude error in the quaternion format corresponding to:

$$q^\varepsilon = \begin{pmatrix} q_o^\varepsilon \\ q_V^\varepsilon \end{pmatrix} = \begin{pmatrix} q_o & -q_V \\ q_V & S_q(q) \end{pmatrix} \begin{pmatrix} q_o^* \\ -q_V^* \end{pmatrix} \tag{5}$$

    Which correponds to $q^\varepsilon = q \star q^{*-1}$, with the operation $\star$ corresponding to the quaternion multiplication of the quaternion algebra (cf. [15]). It can also be written : $q = q^\varepsilon \star q^*$, meaning q^ε is the rotation correction to be performed to bring the object from the theoretical attitude q^* to the real current attitude q. The object is on the trajectory meaning $q = q^*$, if and only if $q^\varepsilon = (\pm 1\ 0\ 0\ 0)^\top$, both values (*double coverage*) representing the same reference attitude corresponding to no rotation correction.

  - $\omega_b^{\varepsilon^*}$ the rotation speed error:

$$\omega_b^{\varepsilon^*} = Q_q(q^*)\omega_b^\varepsilon = Q_q(q^*)(\omega_b - \omega_b^*) \tag{6}$$

    The multiplication of the rotation speed error vector $\omega_b^\varepsilon = \omega_b - \omega_b^*$ by the rotation matrix $Q_q(q^*)$ does not change its norm which is equal to: $||\omega_b^\varepsilon|| = ||\omega_b - \omega_b^*|| = ||\omega_b^{\varepsilon^*}||$. Therefore, when $\omega_b^{\varepsilon^*} = 0$, we get $\omega_b = \omega_b^*$ which corresponds to be on the trajectory for the rotation speed.

- The usual rotation matrix $Q_q(q) \in \mathbb{R}^{3\times 3}$ corresponding to the rotation defined by the quaternion q, such that any vector V_b expressed in the rigid body referential is obtained in the earth referential as follow: $V_e = Q_q(q)V_b$, can be computed with the following relationships:

$$\begin{aligned} Q_q(q) &= \begin{pmatrix} q_V & S_q(q) \end{pmatrix} \begin{pmatrix} q_V^\top \\ S_q(q) \end{pmatrix} \\ &= I_3 + 2q_o S_V(q_V) + 2S_V(q_V)^2 \\ &= I_3 + \sin(\theta) S_V(n) + (1-\cos(\theta)) S_V(n)^2 \end{aligned} \tag{7}$$

The inverse rotation matrix, in the sense of the opposite rotation $-\theta$ around n, corresponding to the unit quaternion "inverse" $q^{-1} = (q_o \; - q_V^\top)^\top$, is therefore directly:

$$Q_q(q)^{-1} = Q_q(q^{-1}) = I_3 - 2q_o S_V(q_V) + 2S_V(q_V)^2$$

- the correction input u defined such that the total torque command C_{act_b} of (1) is equal to:

$$\begin{aligned} C_{act_b} = &\; J_b^* \dot{\omega}_b^* + S_V(\omega_b^*) J_b^* \omega_b^* \\ &+ S_V(\omega_b^*)(2J_b^* - Tr(J_b^*) I_3)\omega_b^\varepsilon \\ &+ S_V(\omega_b^\varepsilon) J_b^* \omega_b^\varepsilon + J_b^* Q_q(q^*)^{-1} u \end{aligned} \tag{8}$$

- the perturbation input $w = w_\Delta + w_{ext}$ such that:
  - w_Δ is the internal perturbation input generated by the model approximation done when computing the theoretical input command $C^*_{act_b}$ with $J_b^* = J_b - \Delta J_b$:

$$\begin{aligned} w_\Delta = Q_q(q^*) J_b^{-1} \big[&- \Delta J_b \dot{\omega}_b^* - S_V(\omega_b^*) \Delta J_b \omega_b^* \\ &- S_V(\omega_b^*)(2\Delta J_b - Tr(\Delta J_b) I_3)\omega_b^\varepsilon \\ &- S_V(\omega_b^\varepsilon) \Delta J_b \omega_b^\varepsilon - \Delta J_b Q_q(q^*)^{-1} u \big] \end{aligned} \tag{9}$$

  - w_{ext} is the external disturbance input due to the external interactions (winds, walls) :

$$w_{ext} = Q_q(q^*) J_b^{-1} C_{ext_b} \tag{10}$$

- the trajectory tracking performance output z_q corresponding to the attitude deviation position q_V^ε.

- the trajectory tracking performance output z_ω corresponding to the rotation speed deviation $\omega_b^{\varepsilon^*}$.

- the measurments $y = x$ considered available through physical sensors measures and data fusion post processing (thus q_o^ε is also available thanks to (4)).

Comment: q_V^ε describes entirely the attitude position of the object except to a sign ambiguity, the *double coverage*. To take away this ambiguity, the following complete state is used for the results analysis:

$$x_c = (sign(q_o^\varepsilon)q_V^{\varepsilon\top} \quad \omega_b^{\varepsilon^*\top})^\top \in \mathbb{R}^6 \tag{11}$$

However, this ambiguity does not affect the definition of the attraction point : the attraction point is the single value $x^o = (0\ 0\ 0\ 0\ 0\ 0)^\top$, meaning the object is exactly on the attitude trajectory when the state is zero.

With this definition of deviation model, we can thus look for a structure of controller that makes the single attraction point x^o asymptotically stable with specific requirements, minimizing the transfer between the perturbation input w and the rotation speed performance output z_ω, and garanteeing that the attitude remains in a tolerance tube of range δ_θ^* around the trajectory ($||z_q|| < 1$) for a worst case maneuver.

3 Attitude tracking control

3.1 Preliminaries

In this paper, the control strategy is to look for a controller that would be the closest to a static state feedback $u = Kx$, with K a constant matrix of $\mathbb{R}^{1\times 6}$, and to take care of the sign ambiguity of q_V^ε by considering continuous multiplication by q_o^ε (which is of the needed sign) of the controller terms inside this static state feedback K that need to flip when the object is backside.

Moreover, as the deviation system (3) appears to be completely homogenous in the three rotation directions of space, it implies that the three components of the attitude and rotation speeds behave identically. That is why we can look for scalar structures of controllers (one scalar gain for each term q_V^ε and $\omega_b^{\varepsilon^*}$) and of corresponding Lyapunov functions without loss of control capacity.

Lemma 1. *The stability property of the attraction point x^o and the output performances related to $||z_q||$ for H with any controller of the form $u = u_r n^\varepsilon$, $u_r \in \mathbb{R}$ a scalar control input and $n^\varepsilon(t)$ the vector defined by $q_V^\varepsilon =$*

$sin(\theta^\varepsilon/2)n^\varepsilon$, are equivalent to the ones of the reduced non linear second order system defined by:

$$
\begin{aligned}
H_r : \dot{x}_r &= A_r(x_r)x_r + B_{w_r}w_r + B_{u_r}u_r \\
z_{q_r} &= C_{q_r}x_r \\
z_{\omega_r} &= C_{\omega_r}x_r \\
y_r &= x_r
\end{aligned}
\tag{12}
$$

With $x_r = (q^\varepsilon_{V_r}\ \ \omega^{\varepsilon^*}_{b_r})^\top = (sin(\theta^\varepsilon/2)\ \ \ \dot{\theta}^\varepsilon)^\top \in \mathbb{R}^2$, $w_r \in \mathbb{R}$ scalar input projection of the original input w on the instantaneous axis n^ε,

$$A_r(x_r) = \begin{pmatrix} 0 & 1/2q^\varepsilon_o \\ 0 & 0 \end{pmatrix}$$

$$B_{w_r} = B_{u_r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C_{q_r} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Comment: H_r corresponds to the projection of H on the direction of the unit vector n^ε, which is the 1-dimension control problem, meaning the control of an object rotating around one single fixed axis. The "complement" system $(H - H_r \otimes n^\varepsilon)$ could be defined specifically to study the influence of the perturbation on the trajectories instantaneous projection on the tangent of the attitude position output z_q equipotentials, meaning trajectories projection orthogonal to the straight direction towards the attraction point.

Proof: Let us notice that the state matrix $A(x)$ of (3) can be decomposed as follow :

$$A(x) = A_r(x) \otimes I_3 + A_V(x) \tag{13}$$

With $\otimes$ the Kronecker product and with:

$$A_V = \begin{pmatrix} 0 & 1/2S_V(q^\varepsilon_V) \\ 0 & 0 \end{pmatrix}$$

The matrix $A_V(x)$ is composed of the single term $1/2S_V(q^\varepsilon_V)$ which corresponds to a vectorial product of q^ε_V with $\omega^{\varepsilon^*}_b$, so the result is orthogonal to q^ε_V and $\omega^{\varepsilon^*}_b$. The matrix $S_V(q^\varepsilon_V)$ is also skew symmetric : $S_V(q^\varepsilon_V)^\top = -S_V(q^\varepsilon_V)$. These properties make the matrix $A_V(x)$ canceling out in the derivative of a quadratic Lyapunov function of the form $x^\top P(x)x$. Indeed, for any $P(x) = P(x)^\top$:

$$x^\top (A_V(x)^\top P(x) + P(x)A_V(x))x = 0$$

Besides, by multiplying on the left side the first line of the differential equation $\dot{q}_V^\varepsilon = 1/2(q_o^\varepsilon + S_V(q_V^\varepsilon))\omega_b^{\varepsilon^*}$ of (3) by $2q_V^{\varepsilon^\top}$, we can notice that the term $S_V(q_V^\varepsilon)$ vanishes giving:

$$d||q_V^\varepsilon||^2/dt = q_o^\varepsilon q_V^{\varepsilon^\top} \omega_b^{\varepsilon^*}$$

So the term $S_V(q_V^\varepsilon)$ does not change the norm of q_V^ε, thus the norm of z_q. It corresponds to a rotation orthogonal to the direction leading to the attraction point, which is consistent with its disappearance in the Lyapunov function derivative.

That's why we can simplify the system (3) to the second order non linear system H_r to study its stability and its output performance related to the $||z_q||$.

3.2 Almost global asymptotical stable controller

Definition 1. *As defined in [16], an autonomous system defined by $\dot{x} = f(x)$, where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is $\mathcal{C}^1$ (enough to ensure the existence and uniqueness of solutions to the initial value problem) and $f(0) = 0$, is almost global asymptotical stable if all the trajectories but a reduced set of zero Lebesgue measure converge asymptotically to the origin.*

Proposition 2. *A non linear state feedback controller of the form:*

$$u(x) = -2k_p q_o^\varepsilon q_V^\varepsilon - k_d \omega_b^{\varepsilon^*} = K(x)x \tag{14}$$

Giving the closed loop non linear system without perturbations ($w = 0$):

$$\begin{aligned} H_{cl} : \dot{x} = A_{cl}(x) &= A(x) + BK(x) \\ &= \begin{pmatrix} 0 & 1/2\ S_q(q^\varepsilon) \\ -2k_p q_o^\varepsilon I_3 & -k_d I_3 \end{pmatrix} \end{aligned} \tag{15}$$

with $K(x) = (-2k_p q_o^\varepsilon I_3 \ \ -k_d I_3)$, $k_p, k_d > 0$ scalar constants, makes this closed loop system almost global asymptotical stable with a compact invariant *set defined by* $\mathcal{W} = \{x^o, \bar{X}\}$, *with* $\bar{X} = \{\bar{x} = (\bar{q}_V^{\varepsilon\top} 0\ 0\ 0)^\top : ||\bar{q}_V^\varepsilon|| = 1\}$.

Comment: H_{cl} is a periodic system for which the exact flipped backside attitudes set $\bar{X}$ from the attraction point $x^o = 0$, rotating around any axis, is a set of unstable equilibrium points with the controller u (this set is equivalent to the condition $\bar{q}_o^\varepsilon = 0$).

This controller structure is similar to a Proportional Derivative controller with respect to the attitude position q_V^ε and the rotation speed $\omega_b^{\varepsilon^*}$, that is why the constants are noted k_p as a "proportional" gain, and k_d as a "derivative" gain. It is close to common quaternion based control approaches developed in other papers, as in [21], that propose similar controllers which are constant linear combinations of the errors of the quaternion vector part and of the rotation speed (mostly after the same feedforward of the inverse kinematics terms (8)). The main difference is in the continuous multiplication by q_o^ε of the attitude deviation q_V^ε. This operation realizes in a continuous way the sign switch of the hybrid control laws developed in [13], taking care of the *double coverage* getting rid of the *unwinding phenomenon*, and without creating conditions for chattering around the unstable equilibrium point. Nonetheless, this controller decreases up to vanish its authority to bring back the object to the attraction point when the object is around the unstable equilibrium points. However these unstable equilibrium points are at the opposite from the tracking tolerance tube in which the controller shall maintain the object (meaning far away from the actuators range sizing). Thus this controller structure could also be a good strategy to deal smoothly with actuators saturation.

The multiplication by 2 of the proportional like term allows to get a nice formulation of the 1-dimension control of a rotating object around one single fixed axis in function of the deviation angle $\theta^\varepsilon = \theta - \theta^*$, which gives the 1-dimension closed loop system (directly equivalent to H_r):

$$\ddot{\theta}^\varepsilon = -k_p \sin(\theta^\varepsilon) - k_d \dot{\theta}^\varepsilon + w_r \qquad (16)$$

Proof: The proof of the propostion 2 is performed using the properties of almost global asymtotic stability as defined in [16], previously mentioned as a dual to Lyapunov's stability theorem [20], which is a derivative of the

Lyapunov stability theorem [9] for systems with several equilibria or invariant sets. We precisely use directly the recent derivative result presented in [7] which is a relaxation of conditions of ISS for multi stable periodic systems (the example in [7] is precisely the demonstration of the stability of the 1-dimension control problem).

Using the Lemma 1, one may define for any $\alpha \in]0;1[$, the class of Lyapunov function $V(x) = x^\top P(x)x$ with the matrix $P(x)$ affine in q_o^ε equal to:

$$P(x) = \begin{pmatrix} 4(k_p + \alpha k_d^2)I_3 & 2\alpha k_d q_o^\varepsilon I_3 \\ 2\alpha k_d q_o^\varepsilon I_3 & I_3 \end{pmatrix} \succ 0 \tag{17}$$

The derivative of the Lyapunov function $V(x)$ for the trajectories of the closed loop system (15) is equal to:

$$\begin{aligned} \dot{V}(x) &= x^\top [A_{cl}(x)^\top P(x) + P(x)A_{cl}(x) + \dot{P}(x)]x \\ &= x^\top \begin{pmatrix} -8\alpha k_p k_d q_o^{\varepsilon 2} I_3 & 0 \\ 0 & -2k_d(1 - \alpha q_o^{\varepsilon 2})I_3 \end{pmatrix} x \\ &\quad - 8\alpha k_d (\dot{q}_o^\varepsilon)^2 < 0 \quad \forall x \notin \mathcal{W} \end{aligned} \tag{18}$$

From the expression of V and $\dot{V}$ and the definition 7 of [7], we can see that V is an ISS Lyapunov function. Therefore it is a practical ISS Leonov function (def. 11 of [7]), which is equivalent to have the closed loop system (15) ISS with respect to the invariant set $\mathcal{W}$ (cf. theorem 14 of [7]), thus the almost global asymptotic stability property.

4 Closed loop system analysis

4.1 Physical representation of the controller

The figure 1 gives a physical assembly representing the action of the "Proportional Derivative" like controller u for the 1-dimension rotation control problem.

The closed loop system with the controller structure u is assimilated to the motion of a blue cylinder of normalized inertia equal to one, which is rotating inside a fixed grey cylinder, sliding in contact with it with an overall friction damping coefficient k_d. A mechanical spring of stiffness k_p attached

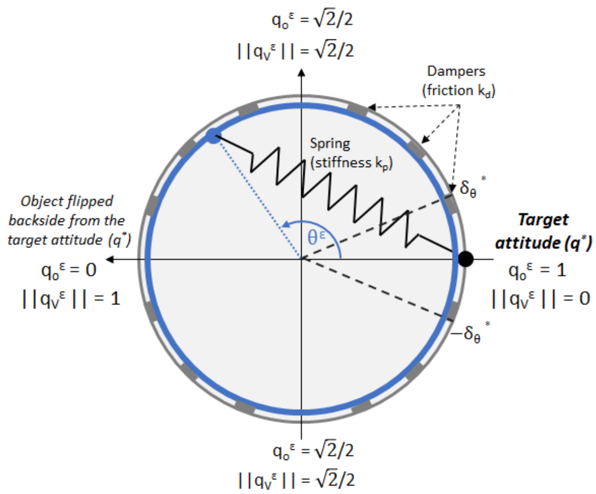

Figure 1: Equivalent physical assembly representing the behaviour of the 1-dimension closed loop system.

on both extremes to the points of each cylinder that have to coincide, brings back the rotating blue cylinder to the required attitude.

4.2 1-dimension closed loop system phase plane

A representation in the phase plane of the 1-dimension closed loop system with the complete state x_c defined by (11) (q_V^ε corrected with the sign of q_o^ε) is given figure 2.

A particular care shall be taken to read the figure 2 : it represents the evolution of the complete state x_c which has a mathematical hybrid behaviour, jumping from $+1$ to -1 or reversely. However this mathematical singularity does not represent any physical discontinuous behaviour: with the definition of the quaternion, a complete rotation is spaned for $q_{V_r}^\varepsilon$ over $[-1;1]$, and the multiplication by $sign(q_o^\varepsilon)$ makes this value jumps when the object crosses backside. It is a way to represent the 2π-periodicity of the rotation motion with the quaternion.

For the theoretical system without perturbations ($w = 0$ in (3)), we can numerically compute the set of initial conditions leading to the unstable

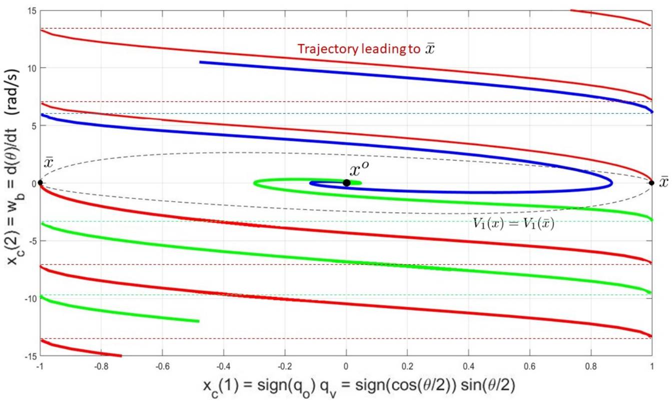

Figure 2: Phase plane for the 1-dimension closed loop system with the proportional derivative like attitude controller, with $k_p = 1$ and $k_d = 1$.

equilibrium attitude set $\bar{X}$. As we are in two dimensions, they correspond to the two symmetric trajectories (the object can turn both sides) integrating backwards the deviation system (3) from the final points $\bar{x}_r = (\pm 1 \ 0)^\top$. These trajectories are drawn in red on the figure 2. The blue and green trajectories are random trajectories converging asymptotically to zero, the stable equilibrium.

5 Controller synthesis

5.1 Performances requirements

The controller shall garantee over a given attitude tolerance tube $\theta^\varepsilon \in [-\delta_\theta^*; \delta_\theta^*]$, equivalent to $q_o^\varepsilon \in [q_o^{\varepsilon *}; 1] = [cos(\delta_\theta^*/2); 1]$, that:

- the overall time constant of the closed loop system is bounded between $[\tau_{min}; \tau_{max}]$ with no oscillations around the trajectory (damping ratio greater than $\sqrt{2}/2$).

- the attitude position remains in the tolerance tube for a worst case sizing maneuver, meaning the output z_q of the input-free closed loop system H_{cl} remains bounded as follow (also called impulse-to-peak performance [6]):

$$\sup_{t \geq 0, ||\alpha||=1} ||z_q(t)|| = \gamma_{IP} < sin(\delta_\theta^*/2) \tag{19}$$

For a given worst rotation speeds initial conditions $\delta\omega_b^{max} > 0$ such that $x(0) = B_0\alpha = \begin{pmatrix} 0 \\ \delta\omega_b^{max} I_3 \end{pmatrix} \alpha$ with $\alpha \in \mathbb{R}^3,\ ||\alpha|| = 1$.

- the feedback interaction between the performance output z_ω and the bias perturbation input w_Δ defined by (9), which has many terms function of ω_b^ε, is minimized. It means to minimize the induced $\mathcal{L}_2$ norm of the closed loop system H_{cl} defined by:

$$||H_{cl}||_2 = \sup_{w \in L_2, ||w||_2 \neq 0} \frac{||z_\omega||_2}{||w||_2} = \gamma_2 \tag{20}$$

5.2 Robust multi-objective synthesis

As we cannot solve directly for the non linear state feedback $K(x) = (-2k_p q_o^\varepsilon I_3 - k_d I_3)$, similarly to a partial linearization, we first solve for a static state feedback $K_r = [k_1\ k_2] \in \mathbb{R}^{1\times 2}$ which bounds the performances of the reduced non linear system H_r defined by the lemma 1 in closed loop with $u_r(t) = K_r x_r(t)$. Thus we look for a static state feedback with a common Lyapunov function over the polytopic set defined by the two vertices:

$$\begin{aligned} A_{r_1} &= A_r(q_o^\varepsilon = 1) \\ A_{r_2} &= A_r(q_o^\varepsilon = q_o^{\varepsilon *}) \end{aligned} \tag{21}$$

embedding the non linear evolution of $A_r(x_r)$ over the tolerance tube $\theta^\varepsilon \in [-\delta_\theta^*; \delta_\theta^*]$, meaning $q_o^\varepsilon \in [q_o^{\varepsilon *}; 1]$. We solve simultaneously for the three previous requirements (5.1) which can be derived into Lyapunov LMI constraints for static state feedback synthesis as developped in [6]. We eventually pick for the non linear state feedback gains:

$$k_p^o = k_1/(2q_o^{\varepsilon *}),\ k_d^o = k_2$$

An optimization of the proportional term can be done by looking for a maximum of $q_o^{\varepsilon^o}$ over the range $[q_o^{\varepsilon *}; 1]$ such that $k_p^o = k_1/(2q_o^{\varepsilon^o})$ makes the original polytopic system (21) verifies independently the two first requirements of paragraph 5.1.

6 Simulations

6.1 Tools and parameters

The Romuloc toolbox [19] for Matlab proposing precoded command to perform multi objective controller synthesis for polytopic systems is used to perform the static state feedback synthesis. The real time simulations are performed with Matlab Simulink.

For the system simulations, the complete input command (8) with the controller (2) is injected in the original non linear system (1) with the following parameters values and hypothesis:

- $J_b^* = diag((0.025 \quad 0.03 \quad 0.02)^\top)$,
- $\delta_\theta^* = 0.635rad(\simeq 36^o = 0.1tr)$, $q_o^\varepsilon \in [0.95; 1]$,
- $[\tau_{min}; \tau_{max}] = [1s; 1.5s]$,
- $\delta\omega_{b_{max}} = 0.5rad/s(\simeq \frac{\delta_\theta^*}{3\tau_{max}})$,
- $(q^*(t), \omega_b^*(t))$ the target trajectory is a rotation at constant speed around the axis $n = (1\ 1 - 1)^\top$ with $\theta^*(t) = t$,
- The real system is biased with $\Delta J_b = diag((0.1 \quad -0.1 \quad 0.1)^\top)J_b^*$ (10% bias),
- The sampling time is $10ms$ and the sensors measures have a $10ms$ delay.
- The object undergoes an impulse deviation from the trajectory at $t = 1s$ such that $x_r(1) = (0 \quad -\delta\omega_{b_{max}})^\top$.

6.2 Results

The multi-objective controller synthesis gives : $k_p^m = 2.976$ and $k_d^m = 3.543$.

The robust multi-objective synthesis (in green on below figures) is compared to the manually tuned linearized reference controller $k_p^o = 0.67$ and $k_d^o = 1.63$, corresponding to the linearized pulsation $\omega_o^o = \sqrt{k_p^o} = 1/\sqrt{\tau_{min}\tau_{max}}$

and damping ratio $\xi^o = \frac{k_d^o}{2\omega_o^o} = 1$ at the attraction point (in purple on below figures).

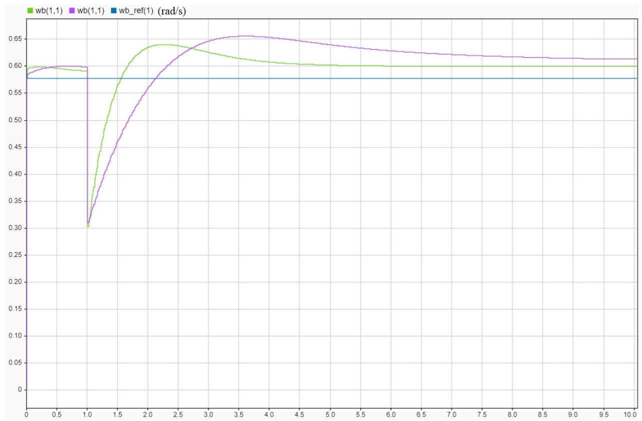

Figure 3: Real time trajectories of the rotation speeds ω_b for the non linear system (1) with previous hypothesis (theoretical trajectory in blue, manually tuned controller in purple, multi objective controller in green).

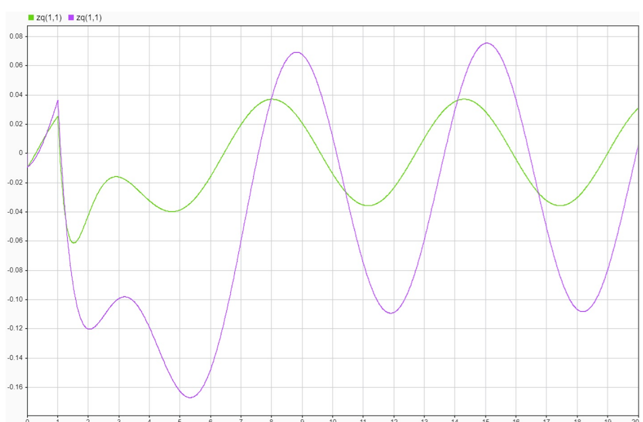

Figure 4: Real time attitude performance output z_q corresponding to previous rotation speed trajectories (manually tuned controller in purple, multi objective controller in green).

7 Conclusion

The results show that a proportional derivative like controller of the form (14) with the non linear complete command (8) allows to perform almost global asymptotical stable tracking of a given feasible attitude trajectory (2) for the original system (1) given multi-objective performances requirements. The chosen performances are arbitrary and many other requirements could be considered (energy input constraint, actuators saturation, perturbation rejection (wind),...). The synthesis procedure takes partially into account the non linearity of the closed loop system, linearizing the proportional like term of the controller itself. In furter research work, an LMI method with S-variables as defined in [6] could be set up to solve structured output feedback design for the original non linear problem. Moreover, the proposed controller has the behavior of a Proportional Derivative controller : a remaining static error appears on the real system with uncertainty. In future work we intend to look for an integral term of the attitude error to be added to the controller to cancel static errors.

References

[1] Hernan Abaunza and Pedro Castillo Garcia. Quadrotor aggressive deployment, using a quaternion-based spherical chattering-free sliding-mode controller. *IEEE Transactions on Aerospace and Electronic Systems*, 56(3):1979–1991, 2020.

[2] Kasey A. Ackerman, Irene M. Gregory, and Naira Hovakimyan. Flight control methods for multirotor uas. *International Conference on Unmanned Aircraft Systems (ICUAS)*, 2019.

[3] V. Alekseev, V. Alexéev, V. Tikhomirov, and Fomine. *Commande optimale*. 1982.

[4] S. Bhat and D. Bernstein. A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems and Control Letters*, 2000.

[5] N. Chaturvedi, A. Sanyal, and N. McClamroch. Rigid-body attitude control. *IEEE Control Systems*, 2011.

[6] Yoshio Ebihara, D. Peaucelle, and D. Arzelier. *S-Variable Approach to LMI-Based Robust Control*. 2015.

[7] Denis Efimov, Johannes Schiffer, Nikita Barabanov, and Romeo Ortega. Relaxing the conditions of ISS for multistable periodic systems. *IFAC-PapersOnLine*, 50(1):7217 – 7222, 2017. 20th IFAC World Congress.

[8] Liming Fan, Hai Huang, Liang Sun, and Kaixing Zhou. Robust attitude control for a rigid-flexible-rigid microsatellite with multiple uncertainties and input saturations. *Aerospace Science and Technology*, 95 105443, 2019.

[9] Wolfgang Hahn. *Stability of Motion*. 1967.

[10] Changran He and Jie Huang. Trajectory tracking and disturbance rejection for quadrotor helicopter by a novel method. *38th Chinese Control Conference*, 2019.

[11] Hao Liu, Xiafu Wang, and Yisheng Zhong. Quaternion-based robust attitude control for uncertain robotic quadrotors. *IEEE TRANSACTIONS ON INDUSTRIAL INFORMATICS*, 11 No. 2, 2015.

[12] Jinhao Liu and Jianying Yang. Spacecraft high accuracy attitude control by quaternion-based nonlinear dynamic inversion. *38th Chinese Control Conference*, 2019.

[13] Christopher G. Mayhew, Ricardo G. Sanfelice, and Andrew R. Teel. Quaternion-based hybrid control for robust global attitude tracking. *IEEE Transactions on Automatic Control*, 56 No. 11, 2011.

[14] Dinu Mihailescu-Stoica, Raul Acuna, and Jürgen Adamy. High performance adaptative attitude control of a quadrotor. *18th European Control Conference*, 2019.

[15] Florian Monteghetti. Quaternions, orientation et mouvement. *Rapport de recherche ISAE-SUPAERO hal-01618257*, 2012.

[16] Pablo Monzon and Rafael Potrie. Local and global aspects of almost global stability. 2006.

[17] Richard M. Murray. Nonlinear control of mechanical systems: A lagrangian perspective. *International Federation of Automatic Control*, 21, pp. 31-42, 1997.

[18] Brad Paden and Ravi Panja. Globally asymptotically stable 'pd+' controller for robot manipulators. *International Journal of Control*, 47 Issue 6, 1988.

[19] Dimitri Peaucelle. Romuloc : Randomized and robust multi-objective control toolbox, 2014.

[20] Anders Rantzer. A dual to Lyapunov's stability theorem. *Systems and Control Letters*, 42(3):161–168, 2001.

[21] Rune Shlanbusch, Antonio Loria, and Per Johan Nicklasson. On the stability and stabilization of quaternion equilibria of rigid bodies. *Automatica*, 48 (2012) 3135-3141, 2012.

[22] Dabo Xu, Jiafan He, Andong Sheng, Zhiyong Chen, and Dan Wang. Robust attitude tracking control of a rigid spacecraft based on nonlinearly controlled quaternions. *11th Asian Control Conference*, 2017.

[23] Tian Ye and Cai Yuanli. Quaternion based sliding mode attitude design for spacecraft. *IEEE 3rd ICCSSE*, 2017.

[24] Zhigang Yu and Guiying Li. Quaternion-based robust sliding mode control for spacecraft attitude tracking. *IEEE 31st Chinese Control and Decision Conference*, 2019.