Multi-Performance State-Feedback for Time-Varying Linear Systems

Thomas Conord, Dimitri Peaucelle

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T. Conord * **D. Peaucelle** *

* *LAAS-CNRS, University of Toulouse, CNRS, UPS, Toulouse, France (e-mail: tconord@laas.fr, peaucelle@laas.fr).*

Abstract: The classical LMI framework for robust multi-objective analysis is extended from time-invariant to time-varying systems. Results concern both input-output performances and bounds on times responses such as the damping ratio. State-feedback is considered using the S-variable approach which allows, at the difference of the Lyapunov Shaping Paradigm, to search for several Lyapunov certificates simultaneously, one for each performance requirement of the multi-objective problem. Results are illustrated by local stabilisation of a non-linear plant with several performance specifications.

Keywords: Time-varying, Non-linear, Performances, Robust, LMIs, S-Variable.

1. INTRODUCTION

The study of dynamical system has been a long time research field. The Lyapunov theory (cf. Lyapunov (1892)) is one of the main initial study that formalized the mathematical principles of stability. These principles have been widely studied to lead to formulations involving state-space matrices constrained by Linear Matrix Inequalities (LMI), as developed in Boyd et al. (1994). These LMI formulations have enabled analysis and controller synthesis frameworks for uncertain Linear Time Invariant (LTI) systems as for example the μ-analysis framework (Duc and Font (2000)), the IQC framework (JoostVeenman et al. (2016); Hu and Seiler (2016)), the S-Variable framework (Ebihara et al. (2015)).

These LMI-based results are not restricted to linear systems and have many derivations for non-linear cases. For example Pettersson and Lennartson (1997) builds an LMI approach to prove the asymptotic stability of some kind of decomposable non-linear systems into sum of affine time-invariant systems. Hyoun-Chul Choi et al. (2008) develops LMI results to demonstrate exponential stability of uncertain Time-Delay Systems. Sadeghi et al. (2016) develops some LMI stability analysis result and robust controller design for some kind of switching systems. While Agulhari et al. (2018) proposes an approach completely based on the transition matrix. The ultimate goal of the research for which the present paper contributes is to go for such results for non-linear systems, with an intermediate step dedicated to time-varying linear systems.

Quite naturally the LMI formalism extended from linear time-invariant to time-varying (LTV) systems leads to Differential Matrix inequalities (DMIs). Many such results are for example cited in Gonçalves et al. (2019), and Seiler et al. (2019) provides appropriate tractable results for finite-horizon analysis of LTV systems. Such results include analysis of stability and input-output performances. As soon as characterization of time-responses is concerned, exponential stability provides information on the decay-rate, see for example necessary and sufficient conditions on the properties of the time-varying state matrix to get the exponential stability in Zhou (2016). This theorem is exploited in Sakai et al. (2020) to develop results for periodic Linear Time-Varying systems, looking directly for solutions of the DMIs taking as assumption that the state matrices is a sum of sine and cosine time functions.

However, these results do not address all performances that may be dealt with using LMIs in the LTI case. The novelty of this paper is the extension from LTI to LTV systems of classical pole location, not only the exponential stability, but also the damping ratio and the natural frequencies, plus, three useful input-output performances analysis results. We provide the DMI formulations for the analysis of these performances and then, for the special case of systems described as included in polytopes we provide LMI conditions for effective state-feedback design. These LMI results are greatly inspired from results in Ebihara et al. (2015) but are not strictly equivalent. We believe these new formulas fit better with the time-varying nature of the considered problem.

The paper is organized as follows. In section 2, we decline the individual DMIs for each dynamic performance analysis. Then in section 3 we explain how we can manage these DMIs constraints as LMIs with constant a Lyapunov certificate. Results assume for simplicity that the time-varying nature of the system is embedded in a polytopic representation. We then derive in section 4 new LMI results for multi-performance state-feedback design. The results are illustrated on a non-linear plant in section 5.

Notation. For a square complex valued matrix M the notation $\{M\}^{\mathcal{H}}$ denotes the Hermitian matrix $\{M\}^{\mathcal{H}} = M + M^*$ where M^* is the transposed conjugate of M. The notation $\star MN$ stands for the Hermitian matrix $\star MN = N^*MN$ and $NM\star = NMN^*$. For two Hermitian matrices M and N, $M \preceq N$ stands for $M - N$ is negative semi-definite. The set $\mathrm{Co}\{A^{[v=1\ldots\bar{v}]}\}$ denotes the polytope defined as the convex hull of the $\bar{v}$ vertices $A^{[v]}$, *ie.* the set of matrices $A(\xi) = \sum_{v=1}^{\bar{v}} \xi_v A^{[v]}$ where $\sum_{v=1}^{\bar{v}} \xi_v = 1$ and $\xi_v \geq 0$.

2. DYNAMIC PERFORMANCE ANALYSIS

In this section we consider continuous-time linear time-varying systems of the type :

$$\begin{cases} \dot{x}(t) = \hat{A}(t)x(t) + B_w(t)w(t) + B_u(t)u(t) \\ z(t) = C(t)x(t) + D_w(t)w(t) \end{cases} \tag{1}$$

in closed-loop with a time-invariant state-feedback $u(t) = Kx(t)$. Let $A(t) = \hat{A}(t) + B_u(t)K$ be the closed-loop state matrix.

Performance analysis results are given in terms of existence of a quadratic time-varying Lyapunov certificate $V(x,t) = x^T(t)P(t)x(t)$ where P is a bounded differentiable matrix valued function of time such that $P(t)$ is symmetric positive-definite for all times t. The set of such functions is denoted $\mathcal{P} = \{P(t) = P^T(t),\ \exists \overline{\lambda} > \underline{\lambda} > 0 : \underline{\lambda} I \preceq P(t) \preceq \overline{\lambda} I\ \forall t \geq 0\}$. Results are formulated as differential matrix inequalities (DMIs) on P.

2.1 Time-responses

Decay rate: The following result provides upper and lower bounds, respectively denoted α_1 and α_2, on the decay rate of time-responses.

Theorem 1. Assume that there exists $P_1 \in \mathcal{P}$, $P_2 \in \mathcal{P}$ and two positive scalars $\lambda_1 > 0$, $\lambda_2 > 0$ such that the following DMIs hold for all $t \in \mathbb{R}_+$:

$$\begin{array}{l} P_1(t) \preceq \lambda_1 I, \quad 2\alpha_1 P_1(t) \preceq \{P_1(t)A(t)\}^{\mathcal{H}} + \dot{P}_1(t), \\ \lambda_2 I \preceq P_2(t), \quad \dot{P}_2(t) + \{P_2(t)A(t)\}^{\mathcal{H}} \preceq 2\alpha_2 P_2(t) \end{array} \tag{2}$$

then the trajectories of $\dot{x}(t) = A(t)x(t)$ are bounded by the following exponentials

$$\beta_1(0)e^{\alpha_1 t} \leq \|x(t)\| \leq \beta_2(0)e^{\alpha_2 t} \tag{3}$$

where $\beta_k^2(0) = \lambda_k^{-1} x(0)^T P_k(0) x(0)$, $k = 1, 2$.

Proof: Let $x(t)$ be the solution of the system for $x(0)$ initial conditions. By congruence, the DMIs (2) imply along trajectories $\dot{x}(t) = A(t)x(t)$ that (dependence in time t is dropped for readability of the formula):

$$\begin{array}{l} 2\alpha_1 x^T P_1 x \leq \{x^T P_1 \dot{x}\}^{\mathcal{H}} + x^T \dot{P}_1 x, \\ x^T \dot{P}_2 x + \{x^T P_2 \dot{x}\}^{\mathcal{H}} \leq 2\alpha_2 x^T P_2 x. \end{array}$$

Let $V_1(t) = x^T(t)P_1(t)x(t)$ and $V_2(x) = x^T(t)P_2(t)x(t)$. These scalar functions hence satisfy the following differential inequalities:

$$2\alpha_1 V_1(t) \leq \dot{V}_1(t) \quad , \quad \dot{V}_2(t) \leq 2\alpha_2 V_2(t).$$

The comparison principle (see **?**) implies:

$$V_1(0)e^{2\alpha_1 t} \leq V_1(t) \quad , \quad V_2(t) \leq V_2(0)e^{2\alpha_2 t}.$$

Since $V_1(t) \leq \lambda_1 \|x(t)\|^2$ and $\lambda_2 \|x(t)\|^2 \leq V_2(t)$ for all times t, the theorem is proved. ∎

If $\alpha_2 < 0$ the Theorem 1 proves exponential stability. The proof follows the classical lines for assessing exponential stability. The next results follows also the same lines but allows to conclude on the damping ratio, which is at our knowledge a new result in the time-varying case.

Damping ratio: The damping of system trajectories is characterized by the ratio between the decay rate and the frequency of oscillatory type responses. This damping ratio is upper bounded by $\tan(\theta)$ in the following theorem. For $\theta = 0$ there is no proved damping. For $\theta = \pi/2$ the damping is infinite meaning that there are no oscillatory trajectories. Oscillatory type responses at frequency ω of time-varying systems are for the following defined as the sum of two terms $x(t) = x_1(t)\cos(\omega t + \phi) + x_2(t)\sin(\omega t + \phi)$ with no other assumption on x_1 and x_2 than being differentiable.

Theorem 2. Let $\theta \in [\,0\,,\ \pi/2\,]$ and assume that there exists $P_3 \in \mathcal{P}$ and a scalar $\lambda_3 > 0$ such that the following DMIs holds for all $t \in \mathbb{R}_+$:

$$\lambda_3 I \preceq P_3(t), \quad \{e^{-j\theta} P_3(t)A(t)\}^{\mathcal{H}} + \cos(\theta)\dot{P}_3(t) \preceq 0 \tag{4}$$

then any oscillatory type response at frequency ω of $\dot{x}(t) = A(t)x(t)$ decays exponentially as follows:

$$\|x(t)\| \leq \beta_3(0)e^{-\omega \tan(\theta) t} \tag{5}$$

where $\beta_3^2(0) = \lambda_3^{-1}(x_1(0)^T P_3(0) x_1(0) + x_2(0)^T P_3(0) x_2(0))$. Hence after n periods of the oscillation the decay is such that $\|x(2n\pi/\omega)\| \leq \beta_3(0)e^{-2n\pi \tan(\theta)}$.

Proof: For simplicity of calculation we shall consider the more general complex valued solution $x(t) = (x_1(t) - jx_2(t))e^{j(\omega t + \phi)}$ the real part of which being the upper defined oscillatory type solution. By congruence, the DMIs (4) imply along trajectories that (recall that $e^{-j(\omega t+\phi)}e^{j(\omega t+\phi)} = 1$):

$$\begin{array}{l} \{e^{-j\theta}(x_1 + jx_2)^T P_3(\dot{x}_1 - j\dot{x}_2 + j\omega x_1 + \omega x_2)\}^{\mathcal{H}} \\ \quad + \cos(\theta)(x_1 + jx_2)^T \dot{P}_3 (x_1 - jx_2) \leq 0 \end{array}$$

After simple calculations this formula reads exactly as:

$$\cos(\theta)\dot{V}_3(t) \leq -2\omega \sin(\theta) V_3(t)$$

where $V_3(t) = x_1^T(t)P_3(t)x_1(t) + x_2^T(t)P_3(t)x_2(t)$. A special important case is when $\theta = \pi/2$. In that case one gets

$$0 \leq -2\omega V_3(t, \omega) \leq 0$$

the right hand side inequality coming from the fact that V_3 is positive definite. This signifies that the only oscillatory response of the system ($\omega \neq 0$) is such that $V_3 \equiv 0$, *ie.* the trivial solution $x \equiv 0$. In all cases the comparison principle implies:

$$V_3(t) \leq V_3(0)e^{-2\omega \tan(\theta) t}.$$

Since $\lambda_3 \|x(t)\|^2 \leq V_3(t)$ for all times t, the theorem is proved. ∎

Natural frequencies: The following theorem proves bounds on the frequencies of oscillatory responses as defined previously.

Theorem 3. Assume that there exists $P_4 \in \mathcal{P}$ and $\overline{\omega} > 0$ such that the following DMIs holds for all $t \in \mathbb{R}_+$:

$$\{-jP_4(t)A(t)\}^{\mathcal{H}} \preceq 2\overline{\omega} P_4(t) \tag{6}$$

then oscillatory type responses of $\dot{x}(t) = A(t)x(t)$ exist only for frequencies $\omega \leq \overline{\omega}$.

Proof: Assume an oscillatory type response $x(t) = (x_1(t) - jx_2(t))e^{j(\omega t + \phi)}$. By congruence, the DMIs (6) imply along trajectories that

$$2\omega V_4(t) \leq 2\overline{\omega} V_4(t)$$

where $V_4(t) = x_1^T(t)P_4(t)x_1(t) + x_2^T(t)P_4(t)x_2(t) \geq 0$. If $\omega > \overline{\omega}$, the only solution is $V_4 \equiv 0$, *ie.* the trivial solution $x_1 \equiv x_2 \equiv 0$. ∎

2.2 Output performance analysis

Impulse-to-Norm performance: The induced Impulse-to-Norm performance evaluates the worst $\mathcal{L}_2$ norm $\bar{\gamma}_2$ of the output $z(t)$ of (1) for a given set of initial conditions $x(0) = B_w(0)\alpha$, $\alpha \in \mathbb{R}^{n_w}$, $||\alpha|| \leq 1$, (or equivalently for zero initial conditions and impulse perturbations $w(t) = \alpha\delta(t)$ where δ is the Dirac impulse):

$$\sup_{||\alpha|| \leq 1} ||z||_2 = \bar{\gamma}_2 \tag{7}$$

Theorem 4. Let $\gamma_2 > 0$, $P_5 \in \mathcal{P}$ such that the following DMIs holds for all $t \in \mathbb{R}_+$:

$$\begin{cases} \{P_5(t)A(t)\}^{\mathcal{H}} + \dot{P}_5(t) + C(t)^T C(t) \preceq 0, \\ B_w^T(0)P_5(0)B_w(0) \preceq \gamma_2^2 I_{n_w} \end{cases} \tag{8}$$

then whatever initial conditions such that $x(0) = B_w(0)\alpha$ with $||\alpha|| \leq 1$ the trajectories of the system (1) are such that $||z||_2 \leq \gamma_2$, *ie.* the induced Impulse-to-Norm performance of the system (1) is bounded by γ_2.

Proof : By congruence on the DMIs (8) we get along the trajectories $\dot{x}(t) = A(t)x(t)$ that:

$$\dot{V}_5(t) + ||z(t)||^2 \leq 0 \quad , \quad V_5(0) \leq \gamma_2^2$$

where $V_5(t) = x^T(t)P_5(t)x(t)$. Integrating the first inequality from 0 to t and combining with the second inequality we get:

$$V_5(t) + \int_0^t ||z(t)||^2 dt \leq V_5(0) \leq \gamma_2^2$$

As $V_5(t) \geq 0$ we get that $||z||_2 \leq \gamma_2$. ∎

Norm-to-Norm performance: The induced Norm-to-Norm performance evaluates the worst induced $\mathcal{L}_2$ norm $\bar{\gamma}_\infty$ between the perturbation input $w(t)$ and the output $z(t)$ of (1), starting from zero initial conditions $x(0) = 0$:

$$\sup_{w \in \mathcal{L}_2, w \neq 0} \frac{||z||_2}{||w||_2} = \bar{\gamma}_\infty \tag{9}$$

The formula is similar to result obtain in the LTI case when applying the KYP Lemma for H_∞ performance (cf. Rantzer (1996)):

Theorem 5. Let $\gamma_\infty > 0$ and assume that there exists $P_6 \in \mathcal{P}$ such that the following DMIs holds for all $t \in \mathbb{R}_+$:

$$\begin{aligned} &\star \begin{pmatrix} \dot{P}_6(t) & P_6(t) \\ P_6(t) & 0 \end{pmatrix} \begin{pmatrix} I_{n_x} & 0 \\ A(t) & B_w(t) \end{pmatrix} \\ &+ \star \begin{pmatrix} I_{n_z} & 0 \\ 0 & -\gamma_\infty^2 I_{n_w} \end{pmatrix} \begin{pmatrix} C(t) & D_w(t) \\ 0 & I_{n_w} \end{pmatrix} \preceq 0 \end{aligned} \tag{10}$$

then the trajectories of the system (1) are such that:

$$\sup_{w \in \mathcal{L}_2, w \neq 0} \frac{||z||_2}{||w||_2} = \bar{\gamma}_\infty \leq \gamma_\infty \tag{11}$$

The induced Norm-to-Norm performance of the system (1) is bounded by γ_∞.

<u>Comment :</u> this result coupled with the theorem 1 is directly equivalent to the one presented in Hu and Seiler (2016) with the IQC approach.

Proof : By congruence on the DMIs (10) we get along trajectories of (1)

$$\dot{V}_6(t) + ||z(t)||^2 \leq \gamma_\infty^2 ||w(t)||^2$$

where $V_6(t) = x^T(t)P_6(t)x(t)$. Integrating this inequality from 0 to t, reminding that $x(0) = 0$ we get:

$$V_6(t) + \int_0^t ||z(t)||^2 dt \leq \gamma_\infty^2 \int_0^t ||w(t)||^2 dt$$

As $V_6(t) \geq 0$, with $t \to \infty$ we get $||z||_2 \leq \gamma_\infty ||w||_2$. ∎

Impulse-to-Peak performance: The induced Impulse-to-Peak performance evaluates the worst instantaneous output range $\bar{\gamma}_{IP}$ of the output $z(t)$ of (1) for a given set of initial conditions $x(0) = B_w(0)\alpha$, $\alpha \in \mathbb{R}^{m_w}$, $||\alpha|| \leq 1$, (or equivalently for zero initial conditions and impulse perturbations $w(t) = \alpha\delta(t)$ where δ is the Dirac impulse):

$$\sup_{t \geq 0, ||\alpha|| = 1} ||z(t)|| = \bar{\gamma}_{IP} \tag{12}$$

Theorem 6. Let $\gamma_{IP} > 0$ and assume that there exists $P_7 \in \mathcal{P}$ such that the following DMIs holds for all $t \in \mathbb{R}_+$:

$$\begin{cases} \{P_7(t)A(t)\}^{\mathcal{H}} + \dot{P}_7(t) \preceq 0 \\ B_w(0)^T P_7(0)B_w(0) \preceq \gamma_{IP}^2 I_{n_w} \\ C(t)^T C(t) \preceq P_7(t) \end{cases} \tag{13}$$

then the trajectories of the system (1) are such that:

$$\sup_{t \geq 0, \alpha \in \mathbb{R}^{n_\alpha}, ||\alpha|| = 1} ||z(t)|| = \bar{\gamma}_{IP} \leq \gamma_{IP} \tag{14}$$

The induced Impulse-to-Peak performance of the system (1) is bounded by γ_{IP}.

Proof : Let $x(t)$ be the solution of the system for the initial conditions $x(0) = B_w\alpha$, $||\alpha|| = 1$, with no perturbation $w(t) = 0$. By congruence on the DMIs (13) we get along the trajectories of (1):

$$\dot{V}_7(t) \leq 0 \quad , \quad V_7(0) \leq \gamma_{IP}^2 \quad , \quad ||z(t)||^2 - V_7(t) \leq 0$$

where $V_7(t) = x^T(t)P_7(t)x(t)$. Integrating the first inequality from 0 to t and combining these three inequalities we get:

$$||z(t)||^2 \leq V_7(t) \leq V_7(0) \leq \gamma_{IP}^2.$$

The inequality holds for all t, hence it holds for the peak value. ∎

2.3 Dual formulations

It is well established that state-feedback design has convex solutions when the upper given formulas, which involve products of the type $P(t)A(t) = P(t)\hat{A}(t) + P(t)B_u(t)K$, are converted to a dual formulation that involve products of the type $A(t)X(t) = \hat{A}(t)X(t) + B_u(t)Y(t)$, where $X(t) = P^{-1}(t)$ and $Y(t) = KX(t)^{-1}$. The latter formulas are easily obtained by congruence. Reminding that $\dot{P} = -X^{-1}\dot{X}X^{-1}$ the dual DMIs are as follows:

- Dual of (2), decay rate:

$$\begin{aligned} &\lambda_1^{-1} I \preceq X_1(t), \quad 2\alpha_1 X_1(t) \preceq \{A(t)X_1(t)\}^{\mathcal{H}} - \dot{X}_1(t), \\ &X_2(t) \preceq \lambda_2^{-1} I, \quad -\dot{X}_2(t) + \{A(t)X_2(t)\}^{\mathcal{H}} \preceq 2\alpha_2 X_2(t). \end{aligned} \tag{15}$$

- Dual of (4), damping ratio:

$$X_3(t) \preceq \lambda_3^{-1} I, \quad \{e^{-j\theta} A(t) X_3(t)\}^{\mathcal{H}} - \cos(\theta)\dot{X}_3(t) \preceq 0. \tag{16}$$

- Dual of (6), frequencies:

$$\{-jA(t)X_4(t)\}^{\mathcal{H}} \preceq 2\varpi X_4(t). \tag{17}$$

- Dual of (8), impulse-to-norm:

$$\begin{cases} \begin{pmatrix} \{A(t)X_5(t)\}^{\mathcal{H}} - \dot{X}_5(t) & X_5(t)C(t)^T \\ C(t)X_5(t) & -I \end{pmatrix} \preceq 0, \\ \gamma_2^{-2} B_w(0)B_w^T(0) \preceq X_5(0) \end{cases} \tag{18}$$

- Dual of (10), norm-to-norm:

$$\begin{pmatrix} I_{n_x} & A(t) \\ 0 & C(t) \end{pmatrix} \begin{pmatrix} -\dot{X}_6(t) & X_6(t) \\ X_6(t) & 0 \end{pmatrix} \star + \begin{pmatrix} B_w(t) & 0 \\ D_w(t) & I_{n_z} \end{pmatrix} \begin{pmatrix} \gamma_\infty^{-2} I_{n_w} & 0 \\ 0 & -I_{n_z} \end{pmatrix} \star \preceq 0. \tag{19}$$

- Dual of (13), imulse-to-peak:

$$\begin{cases} \{A(t)X_7(t)\}^{\mathcal{H}} - \dot{X}_7(t) \preceq 0, \\ \gamma_{IP}^{-2} B_w(0)B_w^T(0) \preceq X_7(0) \\ C(t)X_7(t)C^T(t) \preceq I. \end{cases} \tag{20}$$

3. POLYTOPIC CASE

The DMI formulas from the previous section are not tractable as long as the time-dependence of the data (matrices A, B_w etc.) are not specified and as long as a choice of function is not made for the unknowns P. In case the data and the unknowns are polynomial functions, many techniques can be used as described in Scherer (2006). These could be sum-of-squares techniques Scherer and Hol (2006) which can be coded using YALMIP by Löfberg (2009), or Polya based results that may be coded using ROLMIP by Agulhari et al. (2019). Trigonometric functions of time may as well be considered with similar approaches, see Megretski (2003). For the following, for simplicity of exposure, we consider a simpler case when the data (matrices A, B_w etc.) is assumed to lie in polytopic sets and the derivatives are possibly unbounded.

For the following we assume the state matrices are known to be inside a polytope:

$$\left\{ \begin{pmatrix} \hat{A}(t) & B_u(t) & B_w(t) \\ C(t) & 0 & D_w(t) \end{pmatrix}, t \in \mathbb{R}_+ \right\} \in \mathrm{Co} \left\{ \begin{pmatrix} \hat{A} & B_u & B_w \\ C & 0 & D_w \end{pmatrix}^{[v=1\dots\bar{v}]} \right\} \tag{21}$$

and we shall assume there is no known bound on the time derivatives. Without more knowledge on the system, the only choice is to search for constant Lyapunov certificates P_i ($\dot{P}_i = 0$), or constant X_i ($\dot{X}_i = 0$) in the dual formulas. Under these assumptions, it is easy to notice that the DMIs are LMIs, and these hold for all t if they hold for the whole polytope. Moreover, by convexity arguments one can prove that the LMIs hold for the whole polytope if and only if they hold for the finite number of vertices $v = 1 \dots \bar{v}$. That fact is trivial for all constraints which are affine in the state-space matrices. For other constraints involving products of state-space matrices convexity is yet preserved. Take for example (8) with $\dot{P}_5 = 0$:

$$\{P_5 A(t)\}^{\mathcal{H}} + C(t)^T C(t) \preceq 0.$$

It is equivalent with a Schur complement to:

$$\begin{pmatrix} \{P_5 A(t)\}^{\mathcal{H}} & C(t)^T \\ C(t) & -I \end{pmatrix} \preceq 0$$

which is linear in $A(t)$ and $C(t)$. The same procedure can be applied to all the LMIs (primal or dual) containing products of state matrices, demonstrating their convexity.

The results are valid for any behavior of the state matrices inside the polytope. Therefore, they can directly be extended to non-linear systems where the state-space matrices are functions of the states, as long as trajectories are guaranteed (for examples due to constraints such as saturations) to maintain the system matrices inside the polytope. An alternative, is to prove that for given bounded initial conditions the trajectories shall remain bounded. This statement can be formalized as a robust impulse-to-peak problem: assume that until any time t the state is bounded and the state-spaces are accordingly in a polytope, and prove that the worst case (peak) value at time t^+ still satisfies the constraints (see also Peaucelle et al. (2012)). This usage of the impulse-to-peak performance is illustrated in the numerical example at the end of the paper.

To illustrate the finite number of LMIs on the vertices for the case of constant dual Lyapunov certificate ($\dot{X}_i = 0$), here are the formulas for the time-response performances:

$$2\alpha_1 X_1 \preceq \{A^{[v]} X_1\}^{\mathcal{H}} \quad , \quad \{A^{[v]} X_2\}^{\mathcal{H}} \preceq 2\alpha_2 X_2 \tag{22}$$

$$\{e^{-j\theta} A^{[v]} X_3\}^{\mathcal{H}} \preceq 0 \quad , \quad \{jA^{[v]} X_4\}^{\mathcal{H}} \preceq 2\varpi X_4. \tag{23}$$

All these conditions have the following structure:

$$r_{i1} X_i + \{r_{i2} A^{[v]} X_i\}^{\mathcal{H}} = \begin{pmatrix} I & A^{[v]} \end{pmatrix} R_i \otimes X_i \begin{pmatrix} I \\ A^{[v]T} \end{pmatrix} \preceq 0 \tag{24}$$

with matrices $R_i = \begin{pmatrix} r_{i1} & r_{i2}^* \\ r_{i2} & 0 \end{pmatrix}$ respectively chosen as:

- $r_{11} = 2\alpha_1$, $r_{12} = -1$ for proving that exponential decay rate is greater than α_1;
- $r_{21} = -2\alpha_2$, $r_{22} = 1$ for proving that exponential decay rate is smaller than α_2;
- $r_{31} = 0$, $r_{32} = e^{j\theta}$ for proving that damping ratio is greater than $\tan(\theta)$;
- $r_{41} = -2\varpi$, $r_{42} = -j$ for proving that frequencies are bounded by ϖ.

For simplicity, we shall say that an LTV system $\dot{x}(t) = A(t)x(t)$ is R_i-stable if the LMIs build based on the matrix R_i are satisfied. This definition matches with the definition of pole location for uncertain LTI systems exploited in Ebihara et al. (2015).

4. MULTI-PERFORMANCE STATE-FEEDBACK

Problem 1. Find a state-feedback gain K such that the following $i = 1 \dots \bar{\imath}$ closed-loop configurations of a same system:

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = (\hat{A}_i(t) + B_{u_i}(t)K)x_i(t) + B_{w_i}(t)w(t) \\ z_i(t) = C_i(t)x_i(t) + D_{w_i}(t)w(t) \end{cases} \tag{25}$$

associated to one given specification Π_i chosen among:

- Σ_i is R_i-stable,
- The Impulse-to-Norm of Σ_i is bounded by γ_{2_i},

- The Norm-to-Norm of Σ_i is bounded by γ_{∞_i},
- The Impulse-to-Peak of Σ_i is bounded by γ_{IP_i}

are simultaneously satisfied.

Notice that the specifications are defined for different systems of the same order. Of course a special case is when the matrices are the same for all $i = 1 \dots \bar{\imath}$. But we may also assume that each performance specification Π_i is defined for a variant of a same plant corresponding to different configurations, each configuration evolving in a different polytope with $\bar{v}_i$ vertices as defined by (21).

For solving this problem, a direct extension of the Lyapunov Shaping Paradigm from Chilali et al. (1999), consists in searching for a common Lyapunov certificate $X_i = X$ for all performance specifications and piling up all the matrix inequalities. Doing so, the dual formulations happen to be linear when applying the invertible change of variable $KX = Y$. Indeed one gets in the formulas $A^{[v_i]}X = (\hat{A}^{[v_i]} + B_u^{[v_i]}K)X = \hat{A}^{[v_i]}X + B_u^{[v_i]}Y$. Let $\mathcal{L}_{\Pi_i,\Sigma_i^{[vi]}}(X,Y) \preceq 0$ denote the LMIs in X and Y obtained when choosing among (15), (16), (17), (18), (19), (20) the formula that corresponds to the performance Π_i, replacing the state-space matrices by their value at vertex v_i and taking $X_i = X$, $\dot{X} = 0$ and $KX = Y$.

Theorem 7. If there exist two matrices $X \succ 0$ and Y simultaneously solution of all LMIs $\mathcal{L}_{\Pi_i,\Sigma_i^{[vi]}}(X,Y) \preceq 0$ $i = 1 \dots \bar{\imath}$, then $K = YX^{-1}$ is a solution to Problem 1.

The advantage of this result is that it involves few decision variables. The main drawback is the conservatism due to searching for a common Lyapunov certificate for all performances. An alternative is the S-variable Shaping Paradigm from Ebihara et al. (2015) which allows to search for different Lyapunov certificates, one for each performance specification, but assuming a common S-variable for all constraints. Note that, at the difference of results in Ebihara et al. (2015), the result of Theorem 8 concerns time-varying systems: a common matrix X_i is required for all vertices of the polytope (and hence for all $t \in \mathbb{R}_+$). Without any prior knowledge on the time derivatives or switches of the time-varying state-space matrices it is not possible to search for more advanced time-dependent certificates X_i.

Let $\mathcal{S}_{\Pi_i,\Sigma_i^{[vi]}}(X_i,S,T,A_{oi}) \preceq 0$ denote the matrix inequalities for performance Π_i and system vertex $\Sigma_i^{[vi]}$ when applying the S-variable approach. These formulas are as follows (modified versions of formulas in Ebihara et al. (2015) that do not apply to the time-varying case). In all formulas $M_i^{[v_i]}(S,T) = \hat{A}_i^{[v_i]}S + B_{u_i}^{[v_i]}T$.

- R_i-stability:

$$R_i \otimes X_i \preceq \left\{ \begin{pmatrix} M_i^{[v_i]}(S,T) \\ -S \end{pmatrix} \begin{pmatrix} A_{o_i} \\ -I \end{pmatrix}^* \right\}^{\mathcal{H}} \qquad (26)$$

- Impulse-to-Norm performance bounded by γ_{2_i}:

$$\begin{pmatrix} 0 & X_i & X_i C_i^{[v_i]T} \\ X_i & 0 & 0 \\ C_i^{[v_i]} X_i & 0 & -I \end{pmatrix} \preceq \left\{ \begin{pmatrix} M_i^{[v_i]}(S,T) \\ -S \\ 0 \end{pmatrix} \begin{pmatrix} A_{o_i} \\ -I \\ 0 \end{pmatrix}^* \right\}^{\mathcal{H}}$$
$$\gamma_{2_i}^{-2} B_{wi}^{[v_i]} B_{wi}^{[v_i]T} \preceq X_i \qquad (27)$$

- Norm-to-Norm performance bounded by γ_{∞_i} :

$$\begin{pmatrix} 0 & X_i & X_i C_i^{[v_i]T} \\ X_i & 0 & 0 \\ C_i^{[v_i]} X_i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_{wi}^{[v_i]} & 0 \\ D_i^{[v_i]} & I \end{pmatrix} \begin{pmatrix} \gamma_{\infty_i}^{-2} I & 0 \\ 0 & -I \end{pmatrix} \star \preceq \left\{ \begin{pmatrix} M_i^{[v_i]}(S,T) \\ -S \\ 0 \end{pmatrix} \begin{pmatrix} A_{o_i} \\ -I \\ 0 \end{pmatrix}^* \right\}^{\mathcal{H}} \qquad (28)$$

- Impulse-to-Peak performance bounded by γ_{IP_i}:

$$\begin{pmatrix} 0 & X_i \\ X_i & 0 \end{pmatrix} \preceq \left\{ \begin{pmatrix} M_i^{[v_i]}(S,T) \\ -S \end{pmatrix} \begin{pmatrix} A_{o_i} \\ -I \end{pmatrix}^* \right\}^{\mathcal{H}}$$
$$\gamma_{IP_i}^{-2} B_{w_i}^{[v_i]} B_{w_i}^{[v_i]T} \preceq X_i \qquad (29)$$
$$C_i^{[v_i]} X_i C_i^{[v_i]T} \preceq I$$

Theorem 8. If there exist two matrices S, T and $\bar{\imath}$ matrices $X_i \succ 0$, A_{oi} simultaneously solution of all constraints $\mathcal{S}_{\Pi_i,\Sigma_i^{[vi]}}(X_i,S,T,A_{oi}) \preceq 0$ $i = 1 \dots \bar{\imath}$, then $K = TS^{-1}$ is a solution to Problem 1.

Proof: The demonstration is given only for the first inequality, the other follow readily. Thanks to the invertible change of variable $T = KS$, the constraint (26) reads with the closed-loop state matrix $A_i^{[v_i]} = \hat{A}_i^{[v_i]} + B_{u_i}^{[v_i]}K$ as:

$$R_i \otimes X_i \preceq \left\{ \begin{pmatrix} A_i^{[v_i]} \\ -I \end{pmatrix} S \begin{pmatrix} A_{o_i}^* & -I \end{pmatrix} \right\}^{\mathcal{H}}.$$

By congruence it implies:

$$\begin{pmatrix} I & A_i^{[v_i]} \end{pmatrix} R_i \otimes X_i \begin{pmatrix} I \\ A_i^{[v_i]T} \end{pmatrix} \preceq 0$$

which is the LMI (24). ∎

The open issue with this last theorem is that the constrains are not linear due to the A_{oi} matrices. A strategy is then to choose a priori the A_{o_i} matrices. The following results provide clues for appropriate choices.

Proposition 1. If the system

$$\Sigma_{oi} : \begin{cases} \dot{x}_i(t) = A_{oi}x_i(t) + B_{w_i}(t)w(t) \\ z_i(t) = C_i(t)x_i(t) + D_{w_i}(t)w(t) \end{cases}$$

doesn't pass the analysis test $\mathcal{L}_{\Pi_i,\Sigma_{oi}^{[vi]}}(X_i,0) \preceq 0$, then $\mathcal{S}_{\Pi_i,\Sigma_i^{[vi]}}(X_i,S,T,A_{oi}) \preceq 0$ is infeasible.

Proof : Again the demonstration is given only the first inequality, the other follow readily. Consider the R_i-stability condition (26). By congruence it implies:

$$\begin{pmatrix} I & A_{oi} \end{pmatrix} R_i \otimes X_i \begin{pmatrix} I \\ A_{oi}^* \end{pmatrix} \preceq 0$$

which is the (dual) analysis condition for proving R_i-stability of $\dot{x}_i(t) = A_{oi}x_i(t)$. ∎

This first proposition allows to eliminate general A_{oi} candidates. In the following we give clues for choosing candidates of the form $A_{oi} = -k_i r_{i2}^* I$ where $r_{i2} = 1$ if the performance Π_i is an input-output performance.

Proposition 2. If there exists X, Y are solutions of $\mathcal{L}_{\Pi_i,\Sigma_{oi}^{[vi]}}(X,Y) \preceq 0$, then for a large enough scalar $k_i > 0$ $\mathcal{S}_{\Pi_i,\Sigma_i^{[vi]}}(X,X,Y,-k_i r_{i2}^* I) \preceq 0$ is feasible.

The proof follows from Theorem 2.9 in Ebihara et al. (2015) and is not reproduced here. The important indication from this results is that choosing very large values of k_i shall lead all matrices X_i to be equal ($S = X = X_i$). The S-variable Theorem 8 has then no advantage compared to Theorem 7.

Meanwhile, from Proposition 1, we get for $A_{io} = -k_i r_{i2}^* I$ that

$$(1\ -k_i r_{i2}^*)\, R_i \begin{pmatrix} 1 \\ -k_i r_{i2} \end{pmatrix} = r_{i1} - 2k_i |r_{i2}|^2.$$

The parameter should satisfy $k_i > \frac{r_{i1}}{2|r_{i2}^2|}$. A reasonable one dimensional line search is hence to solve the design problem choosing $k_i = \underline{k}_i(1+\kappa)$ with $\underline{k}_i = \frac{r_{i1}}{2|r_{i2}^2|}$, for $\kappa > 0$. Following the same reasoning for the other performances we get the following heuristic strategy.

Heuristic line-search Solve the LMIs of Theorem 8 with fixed values $A_{oi} = -(1+\kappa)\underline{k_i} r_{i2}^* I$, where

- for the R_i stability: $\underline{k}_i = \frac{r_{i1}}{2|r_{i2}^2|}$,
- for the Impulse-to-Norm bounded by γ_{2_i}:
$$\underline{k}_i = \max_{v_i=1\ldots\bar{v}_i} \left(\frac{1}{2\gamma_{2_i}^2} \lambda_{max} \left(C_i^{[v_i]} B_{o2_i}^{[v_i]} (C_i^{[v_i]} B_{o2_i}^{[v_i]})^T \right) \right)$$
- for the Norm-to-Norm bounded by γ_{∞_i}:
$$\underline{k}_i = \max_{v_i=1\ldots\bar{v}_i} \frac{\lambda_{max}(C_i^{[v_i]} B_{w_i}^{[v_i]})}{\gamma_{\infty_i} - \lambda_{max}(D_{w_i}^{[v_i]})}$$
- for the Impulse-to-Peak bounded by γ_{IP_i} : $\underline{k}_i = 0$.

and search for the best solutions with respect to $\kappa > 0$.

5. ILLUSTRATIVE EXAMPLE

5.1 Application case and tools

We consider the synthesis of a state-feedback for the following nonlinear system, extension with 2 integrators of the reduced attitude deviation tracking model given in the Lemma 1 of Conord and Peaucelle (2021):

$$\begin{aligned} H : \dot{x} &= \hat{A}(x)x + B_w w + B_u u \\ z_q &= C_q x \\ z_\omega &= C_\omega x \end{aligned} \tag{30}$$

with the state $x = (\eta_{V_1}\ \eta_{V_0}\ q_V\ \omega_b)^\top \in \mathbb{R}^4$, the control input $u \in \mathbb{R}$, the perturbation input $w \in \mathbb{R}$, and the state space matrices:

$$\hat{A}(x) = \begin{pmatrix} 0 & q_o^2 & 0 & 0 \\ 0 & 0 & 2q_o & 0 \\ 0 & 0 & 0 & \frac{1}{2}q_o \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_u = B_w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ C_q = (0\ 0\ 1\ 0),\ C_\omega = (0\ 0\ 0\ 1)$$

and the non-linear parameter q_o solution of: $\frac{dq_o}{dt} = -\frac{1}{2} q_V \omega_b$ which also respects the unit norm constraint: $q_o^2 + q_V^2 = 1$.

Specifications: We consider the problem 1 for H with:

- Π_1 : the decay rate greater than $\alpha_1 = -20 rad/s$,
- Π_2 : the decay rate lower than $\alpha_2 = -2 rad/s$,
- Π_3 : the overall damping ratio greater than $\sqrt{2}/2$ ($\tan(\theta) \geq 1$),
- Π_4 : the Norm-to-Norm performance between the perturbation input w and the output z_ω is minimized:
$$Min \sup_{w \in \mathcal{L}_2, w \neq 0} \frac{||z_\omega||_2}{||w||_2}$$
- Π_5 : the induced Impulse-to-Peak performance $|z_q| = |q_V| < \delta q$ for the set of sizing worst case initial conditions $x(0) = B_{w_5}\alpha, B_{w_5} = 10\delta q B_w, \alpha \in \mathbb{R}, |\alpha| = 1$:
$$\sup_{t \geq 0, \alpha_q \in \mathbb{R}, |\alpha_q|=1} |z_q(t)| < \delta q$$

Polytope: The sizing Impulse-to-Peak specification Π_5 is the requirement of the operational domain evolution of the system. It leads to the bounds of evolution of $q_o \in [q_o^*; 1]$ with $q_o^* = \sqrt{1 - \delta q^2}$, which gives the possible non-linear values of the state matrix $A(x)$. This set of values can be embedded in a polytope defined by the three following vertices:

$\hat{A}^{[1]} = \hat{A}(q_o = 1)$, $\hat{A}^{[2]} = \hat{A}(q_o = q_o^*)$, and

$$\hat{A}^{[3]} = \begin{pmatrix} 0 & q_o^* & 0 & 0 \\ 0 & 0 & 2q_{o_2}^* & 0 \\ 0 & 0 & 0 & 1/2q_{o_2}^* \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } q_{o_2}^* = 1/2(1 + q_o^*).$$

The Romuloc toolbox of Peaucelle (2014) for Matlab, proposing precoded command to perform multi objective controller synthesis for polytopic systems, can be used to solve directly the LMIs of theorem 7. The LMIs of theorem 8 involving S-Variables are coded manually using the Yalmip toolbox and the solver SDPT3 with Matlab.

5.2 Results

The results are computed for $\delta q = 0.4$ which gives $q_o^* = 0.9165$ and $q_{o_2}^* = 1/2(q_o^* + 1) = 0.9583$, and the controllers:

- for the linearized LTI system A_1, the reference manually computed state-feedback for a single decay rate fixed at $\alpha_o = -\sqrt{|\alpha_1|.|\alpha_2|} = -6.3 rad/s$ and a damping ratio equal to 1:
$$K_o = [-1600\ \ -1012\ \ -480\ \ -25.3]$$
Norm-to-Norm performance: $\gamma_{\infty_o} = 0.0031$.
- for the theorem 7:
$$K_X = [-1954\ \ -1388\ \ -768.5\ \ -31.1]$$
Norm-to-Norm performance: $\gamma_{\infty_X} = 0.0017$.
- for the theorem 8 with an the Heuristic line search for A_{o_4} performed with an initial gess $\underline{k}_4 = -1/(\gamma_{\infty_X})$ (the S-variable approach should improve the result of theorem 7):
$$K_S = [-196.6\ \ -429.9\ \ -1008.4\ \ -59]$$
Norm-to-Norm performance: $\gamma_{\infty_S} = 0.0011$.

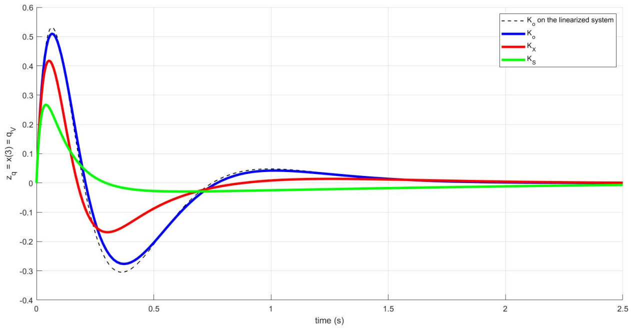

Fig. 1. Non linear closed-loop system $x(3) = q_V$ free response from $x(0) = (0\ 0\ 0\ 100\delta q)^T$ with the different controllers.

6. CONCLUSION

The LMI approach presented in this paper enables to make the synthesis of time-invariant state-feedback controllers for time-varying systems that lie in polytopic sets, with their derivatives possibly unbounded. Feasible solutions depend on the polytopic set chosen to embed the original non-linear or time-varying system: obviously if the polytopic set is too wide, there may not exist a feasible time-invariant state-feedback solution for any specification. Unfortunately, even if state-feedback gains satisfying all performance constraints exist, the LMIs may not provide any solution due to conservatism. The S-variable result being potentially less conservative can give solutions (as demonstrated on the example) when the more conservative Lyapunov Shaping Paradigm formulas fail. The drawback of the S-variable result lies in the need of choosing a priori some design parameters. To cope with this issue, clear guidelines are proposed for a comprehensive choice of these parameters. The strategy deserves to be validated on more advanced examples which will be considered in future studies. Further, even less conservative solutions may derived by more advanced treatment of the original DMI formulations of the multi-performance problem.

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