Insight into the stability analysis of the reaction-diffusion equation interconnected with a finite-dimensional system taking support on Legendre orthogonal basis

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Abstract The stability analysis of the reaction-diffusion subject to dynamic boundary conditions is not straightforward. This chapter proposes a linear matrix inequality criterion which ensures the stability of such infinite-dimensional system. By the use of Fourier-Legendre series, the Lyapunov functional is split into an augmented finite-dimensional state including within it the first Fourier-Legendre coefficients and the residual part. A link between this modelling and Padé approximation is briefly highlighted. Then, from Bessel and Wirtinger inequalities applied to the Fourier-Legendre remainder and using its orthogonality properties, a sufficient condition of stability expressed in terms of linear matrix inequalities is obtained. This efficient and scalable stability condition is finally performed on examples.

1 Introduction

The wide class of infinite-dimensional systems [6, 7, 10] generates issues to be analyzed numerically. In most of the cases, when the eigenvalue decomposition of the infinite-dimensional operator cannot be given analytically, the stability properties of these systems remain unknown. For instance, the stability analysis of a system coupled with a partial differential equation is tough task, which has been infrequently studied. Most of recent studies bypass the problem by focusing on the stabilization of such infinite-dimensional systems thanks to the design of infinite-dimensional control laws [1, 22].

In the field of numerical analysis, researchers approximate the solutions from spacial discretization [17] or the tau method [12]. Under dissipativity conditions [13], convergence properties of the numerical schemes emerge. But, they never deal with

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the stability of the original system. Therefore, to rule on the stability of interconnected ordinary-partial differential equations, input-to-state [15, 19, 20] or Lyapunov [21, 26] approaches are privileged. For generic cases, quadratic constraints and complete Lyapunov functionals lead to criteria solved be semi-definite programming [8, 9, 25]. To refine, adjust and better understand this generic results, works have been pursued on the model and the inequalities involved in the sufficient condition of stability. For instance, with transport [23], heat [5] or wave [4] equations, an augmented system enriched by Legendre polynomials coefficients of the partial-differential part can be built. Applying Bessel-Legendre inequality, hierarchical stability results are expressed in terms of linear matrix inequalities and dedicated to each equation. Concerning the reaction-diffusion equation, the work is slighty modified since the Wirtinger inequality has to be invoked. In this chapter, one aims at simplifying the linear matrix inequality condition obtained enlightening the Fourier-Legendre series, as done in [2] with other boundary conditions.
To sum up, a numerical approach to state on the stability of an ordinary differential equation interconnected with a reaction-diffusion equation is proposed. The novelty of this contribution comes from the model transformation which eases the expression and understanding of the stability analysis. Indeed, by defining signals derived from Fourier-Legendre remainder, the Bessel-Legendre and the first Wirtinger inequalities can be rewritten in a nice formalism. Around these signals, the dynamical model can be constructed and related to Padé $(n-1|n)$ approximant. It includes the extended finite-dimensional based on the first Fourier-Legendre polynomials coefficients. Taking a quadratic Lyapunov function based on this augmented system, a tight linear matrix inequality condition of stability is obtained. Numerical results are finally performed to show the effectiveness of the approximation and of the sufficient stability criterion.

Notations : In this paper, the set of natural numbers, real numbers, real positive numbers, matrices of size $n \times m$ and of symmetric positive definite matrices of size n are respectively denoted $\mathbb{N}$, $\mathbb{R}$ $\mathbb{R}_{\geq 0}$, $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n_+$, respectively. The identity matrix of dimension n is denoted by I_n and $\mathrm{diag}(d_0, ..., d_n)$ stands for the diagonal matrix whose coefficients are $(d_0, ..., d_n)$. For any matrix M, $M^{i,j}$ refers to the coefficient located on the i^{th} row and j^{th} column. For any square matrix M, the transpose matrix is denoted $M^\top$, $\mathcal{H}(M) = M + M^\top$ and $M \succ 0$ means that M is symmetric positive definite. For any square matrix M, $\sigma(M)$ denotes the spectrum of M. Furthermore, if M is symmetric, $\underline{\sigma}(M)$ and $\bar{\sigma}(M)$ stands for its lower and larger eigenvalues, respectively. For any analytic function G_1 and G_2, $G_1(s) = \underset{s\to\lambda}{O}\big(G_2(s)\big)$ means that the ratio $\frac{G_1}{G_2}(s)$ is finite for s tends to $\lambda \in \mathbb{R}$. Moreover, the space of square integrable functions $\mathcal{L}^2(a, b; \mathbb{R})$ is associated to the scalar product $\langle z_1|z_2\rangle = \int_a^b z_1(\theta)z_2(\theta)\mathrm{d}\theta$ and the induced norm $\|z\|^2 = \int_a^b z^2(\theta)\mathrm{d}\theta$. Set $\mathcal{H}^1(a, b; \mathbb{R})$ stands for the set of functions z, such that z and $\partial_\theta z$ are in $\mathcal{L}^2(a, b; \mathbb{R})$. With a light abuse of notations, the notation for inner product $\langle z_1|z_2\rangle$ will be used when z_1 and z_2 are vector functions.

2 Presentation of the system

2.1 Interconnected system

Consider the following system composed of an ordinary differential equation interconnected with a reaction-diffusion partial differential equation with cross type boundary conditions

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(t,1), & \forall t \in \mathbb{R}_{\geq 0}, & \text{(1a)} \\ \partial_t z(t,\theta) = (\delta\partial_{\theta\theta} + \lambda)z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_{\geq 0} \times (0,1), & \text{(1b)} \\ \begin{bmatrix} z(t,0) \\ \partial_\theta z(t,1) \end{bmatrix} = \begin{bmatrix} 0 \\ Cx(t) \end{bmatrix}, & \forall t \in \mathbb{R}_{\geq 0}, & \text{(1c)} \\ (x(0), z(0,\cdot)) = (x_0, z_0). & & \text{(1d)} \end{cases}$$

Coefficients $\delta > 0$, $\lambda \in \mathbb{R}$, matrix $A \in \mathbb{R}^{n_x \times n_x}$ and vectors $B, C^\top \in \mathbb{R}^{n_x}$ are supposed to be constant and known.

2.2 Existence and uniqueness of solutions

As a first step and before studying stability of such class of system, it is important to show that system (1) is well-posed despite unbounded input and output operators applied on the partial differential equation [18]. This is formulated in the next proposition.

Proposition 1 *Assuming that (x_0, z_0) belongs to $\mathbb{R}^{n_x} \times \mathcal{L}^2(0,1;\mathbb{R})$, system* (1) *admits a continuous and unique solution $(x, z) \in \mathbb{R}^{n_x} \times \mathcal{L}^2(0,1;\mathbb{R})$.*

Proof Let us define the energy of system (1) by $\mathcal{E}(t) = x^T(t)x(t) + (2\delta)^{-1}\|z(t)\|^2$. Taking its derivatives, calculations lead to

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = {} & x^T(t)(A^T + A)x(t) + 2x^T(t)Bz(t,1) + \tfrac{\lambda}{\delta}\|z(t)\|^2 \\ & -\|\partial_\theta z(t)\|^2 + x^T(t)C^T z(t,1). \end{aligned}$$

Getting rid of the cross terms by application of Young's inequality, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leq x^T(t)\left(\mathcal{H}(A) + 4BB^T + C^T C\right)x(t) + \frac{\lambda}{\delta}\|z(t)\|^2 - \|\partial_\theta z(t)\|^2 + \frac{1}{2}z(t,1)^2.$$

However, from Jensen's inequality, one gets to

$$\|\partial_\theta z(t)\|^2 \geq z(t,1)^2, \tag{2}$$

which simplifies the previous inequality on $\dot{\mathcal{E}}(t)$, leading to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leq K\mathcal{E}(t) - \tfrac{1}{2}\|\partial_\theta z(t)\|^2 \leq K\mathcal{E}(t),$$

where K is a generic constant depending on $A, B, C, \lambda, \epsilon$.
From Grönwall's inequality, there exists a unique solution (x, z) in the continuous space from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^{n_x} \times \mathcal{L}^2(a, b; \mathbb{R})$. ∎

Remark 1 In Section 5, to deal with stability of system (1), a Lyapunov functional is chosen to be equivalent to the energy $\mathcal{E}(t)$ in the sense of the $\mathbb{R}^{n_x} \times \mathcal{L}^2(0, 1; \mathbb{R})$. In parallel with the calculations carried out here, the positive term in $\|z\|$ is upper bounded by Wirtinger inequality and Jensen's inequality (2) is improved thanks to Bessel-Legendre inequality.

2.3 Equilibrium point

As a second step, it is also important to characterize the equilibrium of (1). More particularly, one has to understand under which conditions, system (1) admits a unique equilibrium. This is formulated below.

Proposition 2 *System* (1) *admits a unique equilibrium* $(0, 0)$ *if and only if matrix* Ω *given by* (3) *has a full rank.*

$$\Omega = \begin{cases} \begin{bmatrix} A & B\sinh(\tilde{\lambda}) \\ C & -\tilde{\lambda}\cosh(\tilde{\lambda}) \end{bmatrix}, & \text{if } \lambda < 0, \\ \begin{bmatrix} A & B \\ C & -1 \end{bmatrix}, & \text{if } \lambda = 0, \\ \begin{bmatrix} A & B\sin(\tilde{\lambda}) \\ C & -\tilde{\lambda}\cos(\tilde{\lambda}) \end{bmatrix}, & \text{if } \lambda > 0. \end{cases} \tag{3}$$

Proof Let $(\bar{x}, \bar{z})$ be an equilibrium of system (1), meaning that the following relations hold.

$$\begin{cases} A\bar{x} + B\bar{z}(1) = 0, & (4a) \\ (\delta\partial_{\theta\theta} + \lambda)\bar{z}(\theta) = 0, & (4b) \\ \bar{z}(0) = 0, & (4c) \\ \partial_\theta \bar{z}(1) = C\bar{x}. & (4d) \end{cases}$$

By integration of the differential equation (4b) and under condition (4c), one obtains

$$\bar{z}(\theta) = \begin{cases} \gamma\sinh(\tilde{\lambda}\theta), & \text{if } \lambda < 0, \\ \gamma\theta, & \text{if } \lambda = 0, \\ \gamma\sin(\tilde{\lambda}\theta), & \text{if } \lambda > 0, \end{cases}$$

where γ in $\mathbb{R}$ to be fixed and $\tilde{\lambda} = \sqrt{|\lambda|/\delta}$. By re-injecting this expression into (4a) and (4d), it yields $\Omega\left[\begin{smallmatrix} \bar{x} \\ \gamma \end{smallmatrix}\right] = 0$ with Ω given by (3). Hence, system (4) admits a unique solution leading to the equilibrium $(\bar{x}, \bar{z}) = (0, 0)$ if and only if $\det(\Omega) \neq 0$ which is equivalent to have Ω a full matrix. ∎

2.4 Expected results

Under the conditions stated above, the aim of this paper is to state accurately on the stability of the equilibrium point $(0, 0)$ using the linear matrix inequality framework. More specifically the following results will be obtained:

- The main result is the proposition of a scalable stability analysis for this coupled ordinary-partial differential equations based on Lyapunov arguments. The analysis is performed thanks to an accurate Lyapunov functional, which is build using Legendre polynomials. The calculations developments are highly inspired by [2] and have been simplified thanks to the consideration of easier boundary condition applied to the reaction-diffusion equation.
- Contrary to a previous study provided in [5], the analysis will be performed through the introduction of the remainder of Legendre-Fourier series, allowing to simplify some technical aspects. Thanks to this remainder, it is possible to rewrite the Wirtinger and the Bessel-Legendre inequalities in a simpler manner compared to the formulation presented in [5].
- Moreover, thanks to the introduction of an augmented system, a necessary condition for the stability condition emerges: a certain matrix $\mathbf{A}_n$ has to be stable. In Section 4, as a subsidiary result, it is shown that this matrix is the Redheffer product between a realization of the Padé $(n-1|n)$ approximant of the reaction-diffusion transfer function and the finite-dimensional system $\left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right]$.

3 Fourier-Legendre remainder and inequalities

Based on the nature of the boundary conditions of the reaction-diffusion (1c), we consider a function z in $\mathcal{L}^2(0, 1; \mathbb{R})$ such as $z(0) = 0$. To construct a complete description of this function z, only odd polynomials on an extended interval $[-1, 1]$ need to be used. Then, after recalling the basics of Legendre polynomials and some of their properties, this section provides the definition of Fourier-Legendre coefficients and reaminder on the interval $[-1, 1]$.
In a last step, several relevant inequalities are presented.

3.1 Fourier-Legendre coefficients and remainder

Legendre polynomials, denoted as l_k for any positive integer k, are given by

$$l_k(\theta) = \sum_{i=0}^{k} (-1)^i \frac{(k+i)!}{(i!)^2 (k-i)!} \left(\frac{1-\theta}{2}\right)^i, \ \forall \theta \in [-1, 1]. \tag{5}$$

The orthogonal family $\{l_k\}_{k\in\mathbb{N}}$ spans $\mathcal{L}^2(-1,1;\mathbb{R})$. For writing comfort, let introduce the notation ℓ_n for any $n \in \mathbb{N}$, which gathers the n first odd Legendre polynomials in vector formulation, that is

$$\ell_n(\theta) = \begin{bmatrix} l_1(\theta) \; l_3(\theta) \; \dots \; l_{2n-1}(\theta) \end{bmatrix}^\top \in \mathbb{R}^n. \tag{6}$$

Recall also some important properties of Legendre polynomials [11], that will be useful along the paper, that are

$$\langle \ell_n | \ell_n \rangle = \mathcal{I}_n^{-1}, \tag{7a}$$
$$\ell_n''(\theta) = \mathcal{L}_n \ell_n(\theta), \quad \forall \theta \in [-1,1], \tag{7b}$$
$$\ell_n(-\theta) = -\ell_n(\theta), \quad \forall \theta \in [0,1], \tag{7c}$$

where matrices $\mathcal{L}_n$ and $\mathcal{I}_n$ are square matrices of dimension n, given by

$$\begin{aligned} \mathcal{I}_n &= \frac{1}{2}\mathrm{diag}(3,\dots,4n-1), \\ \mathcal{L}_n^{i,j} &= (4j-1)\sum_{p=j}^{i-1}(4p+1), \quad \forall i,j \in \{1,\dots,n\}. \end{aligned} \tag{8}$$

Functions ℓ_n can be easily evaluated at $\theta \in \{0,1\}$ and is given by

$$\ell_n(1) = [\, 1 \; 1 \; \dots \; 1 \,]^\top \in \mathbb{R}^n, \quad \ell_n(0) = [\, 0 \; 0 \; \dots \; 0 \,]^\top \in \mathbb{R}^n, \tag{9a}$$
$$\ell_n'(1) = [\, 1 \; 6 \; \dots \; n(2n-1) \,]^\top \in \mathbb{R}^n. \tag{9b}$$

As stated in the previous section, we will define here the main features of this paper in the following definition.

Definition 1 For any function $z \in \mathcal{L}^2(0,1;\mathbb{R})$ such that $z(0) = 0$ and for integer n in $\mathbb{N}$, we define

- the state z on the extended interval $[-1,1]$ by

$$z(\theta) = -z(-\theta), \quad \forall \theta \in [-1,0], \tag{10}$$

- the n first Fourier-Legendre coefficients of z as

$$\zeta_n = \begin{bmatrix} \int_{-1}^{1} l_1(\theta) z(\theta) \mathrm{d}\theta \\ \vdots \\ \int_{-1}^{1} l_{2n-1}(\theta) z(\theta) \mathrm{d}\theta \end{bmatrix} = \langle \ell_n | z \rangle \in \mathbb{R}^n, \tag{11}$$

- the associated Fourier-Legendre remainder of z at order n as

$$w_n(\theta) = z(\theta) - \ell_n^\top(\theta)\mathcal{I}_n \zeta_n, \quad \forall \theta \in [-1,1]. \tag{12}$$

It is worth noting that w_n is in $\mathcal{L}^2(-1,1;\mathbb{R})$ and that the norm and scalar product used in the sequel are considered in the interval $[-1,1]$. The main interest for introducing this remainder is stated in the two following lemmas.

Lemma 1 *For any n in $\mathbb{N}$, the Fourier-Legendre remainder w_n given by* (12) *verifies*

- $\langle l_k|w_n\rangle = 0$*, for any integer $k \in \{0,\dots,2n\}$, which means in other words that w_n is orthogonal to the $2n+1$ first Legendre polynomials,*
- $w_n(0) = 0$*, which guarantee that the boundary condition at $\theta = 0$ satisfied by z is maintained for w_n.*

Proof By symmetry of all even Legendre polynomials, one obtains directly that $\langle l_{2k}|w_n\rangle = 0$, for all positive integers k. Moreover, thanks to the orthogonality (7a) of the Legendre polynomials, re-injecting the definition of w_n into $\langle \ell_n|w_n\rangle$ yields

$$\langle \ell_n|w_n\rangle = \langle \ell_n|z\rangle - \langle \ell_n|\ell_n\rangle\, \mathcal{I}_n\, \langle \ell_n|z\rangle = \langle \ell_n|z\rangle - \langle \ell_n|z\rangle = 0,$$

which lead to the orthogonality on the n first odd Legendre polynomials. Lastly, thanks to the value at $\theta = 0$ given by (9a), we have

$$w_n(0) = z(0) - \ell_n^\top(0)\mathcal{I}_n\zeta_n = z(0) = 0.$$

which concludes the proof. ■

In the next subsections, Bessel and Wirtinger inequalities are rewritten in an adequate manner to be consistent with the Fourier-Legendre remainder introduced above. The forthcoming Lyapunov analysis could then be pursued on an augmented system which takes aside the Fourier-Legendre remainder w_n of the signal z.

3.2 Bessel inequalities

Let us first recall the Bessel-Legendre inequality.

Lemma 2 *For any function $z \in \mathcal{L}^2(-1,1;\mathbb{R})$ and for any integer n in $\mathbb{N}$, the following inequality holds:*

$$\|z\|^2 \geq \sum_{k=0}^{n} \left(\frac{2k+1}{2}\right) \langle l_k|z\rangle^2 . \tag{13}$$

Proof The proof can be found in [2, 5]. It is directly derived from the positivity of the $\mathcal{L}^2(-1,1;\mathbb{R})$ norm of the Fourier-Legendre remainder of the function z. ■

The application of Lemma 2 on Fourier-Legendre remainder yields $\|w_n\|^2 \geq 0$, since the remainder is orthogonal to the first Legendre polynomials, i.e. $\langle \ell_n|w_n\rangle = 0$. This means that the information encapsulated in the Bessel inequality is already included in the remainder.

The interest of using the remainder is related to the formulation of this inequality when it is applied to its derivatives $\|w_n'\|$ that is presented in the next lemma.

Lemma 3 *For any function z in $\mathcal{H}^1(0,1;\mathbb{R})$ such that $z(0)=0$ and any integer n in $\mathbb{N}$, the derivatives of the remainder w_n given by* (12) *verifies*

$$\|w_n'\|^2 \geq \kappa_n^B w_n^2(1), \quad \textit{with} \quad \kappa_n^B = 2(n+1)(2n+1). \tag{14}$$

Proof Thanks to the Bessel-Legendre inequality at order $2n+1$, the following inequality holds

$$\|w_n'\|^2 \geq \sum_{k=0}^{2n+1} \left(\frac{2k+1}{2}\right) \left\langle l_k \middle| w_n' \right\rangle^2. \tag{15}$$

In addition, performing an integration by parts yields

$$\left\langle l_k \middle| w_n' \right\rangle = \left(l_k(1) + l_k(-1)\right) w_n(1) - \left\langle l_k' \middle| w_n \right\rangle.$$

Then, we recall that w_n is the remainder of the Fourier-Legendre series, which is consequently orthogonal to the $2n+1$ first Legendre polynomials. Therefore, the last term of the previous equality is zero so that

$$\|w_n'\|^2 \geq \left(2\sum_{p=0}^{n}(4p+1)\right) w_n^2(1).$$

To complete the proof, we have

$$\sum_{k=0}^{n}(4p+1) = 4\frac{n(n+1)}{2} + (n+1) = (n+1)(2n+1).$$

which yields the result. ∎

Remark 2 It is worth noticing that the error made in this new Bessel-inequality is equal to the $\mathcal{L}^2(-1,1;\mathbb{R})$ norm of the Fourier-Legendre remainder of z'. Then, this error converges to zero since z' belongs to $\mathcal{L}^2(0,1;\mathbb{R})$.

It is important to note that the previous lemma allows to express a lower bound on the derivative with respect to θ of the Fourier-Legendre series, which only depends on the evaluation of w_n at the boundary $\theta = 1$. This bound does not depends on the first Fourier-Legendre coefficients, since we have chosen to consider the remainder only, which is orthogonal to the $2n+1$ first Legendre polynomial. This will simplify many technical calculations in the next developments.

3.3 Modified Wirtinger's inequality

In the literature [14], Wirtinger's inequalities refer to inequalities which estimate the integral of the derivative function with the help of the integral of the function. These inequalities have been widely used in the context of analysis, control and observation of time-delay and reaction-diffusion systems [24]. In this paper, one proposes to use Wirtinger's inequality of the first type, stated as follows.

Lemma 4 *For any function $\tilde{z}$ in $\mathcal{H}^1(-1,1;\mathbb{R})$ such that $\tilde{z}(-1) = \tilde{z}(1)$ and that $\langle \tilde{z}|1\rangle = 0$, the following inequality holds:*

$$\|\tilde{z}'\| \geq \pi\|\tilde{z}\|. \tag{16}$$

Proof The proof comes along the arguments provided in [14]. Relying on the Fourier series of $\tilde{z}$ given by $\sum_{k=0}^{\infty} c_k e^{jk\pi\theta}$ with $c_k = \left\langle \tilde{z}(\theta)\middle| e^{-jk\pi\theta}\right\rangle$, it follows $\|\tilde{z}\|^2 = \sum_{k=0}^{\infty} c_k^2$. Moreover, taking also the Fourier series of $\tilde{z}'(\theta)$ as $\sum_{k=0}^{\infty} d_k e^{jk\pi\theta}$ with $d_k = \left\langle \tilde{z}'(\theta)\middle| e^{jk\pi\theta}\right\rangle$, we also have $\|\tilde{z}'\|^2 = \sum_{k=0}^{\infty} d_k^2$. The result is then obtained by integration by parts, noticing that $d_k = jk\pi c_k$. ■

The next lemma is an application of the previous Wirtinger inequality to the Fourier-Legendre remainder w_n given by (12).

Lemma 5 *For any function z in $\mathcal{H}^1(0,1;\mathbb{R})$ and for any n in $\mathbb{N}$, the Fourier-Legendre remainder w_n given by* (12) *verifies*

$$\|w_n'\|^2 - \pi^2\|w_n\|^2 \geq \kappa_n^W w_n^2(1), \quad \textit{with} \quad \kappa_n^W = 2n(2n-1) + \frac{2\pi^2}{4n-1}. \tag{17}$$

Proof In order to apply Lemma 4, let us introduce function $\tilde{w}_n$, defined by

$$\tilde{w}_n(\theta) = w_n(\theta) - l_{2n-1}(\theta) w_n(1), \quad \forall \theta \in [-1,1]. \tag{18}$$

First, one has to verify the assumptions of the Wirtinger inequality of the first kind, that is $\tilde{w}_n(-1) = \tilde{w}_n(1) = 0$ and $\langle \tilde{w}_n|1\rangle = 0$. Recalling that $l_{2n-1}(1) = 1$ and that $l_{2n-1}(-1) = -1$, we have

$$\tilde{w}_n(1) = w_n(1) - w_n(1) = 0, \quad \tilde{w}_n(-1) = w_n(-1) + w_n(1) = 0.$$

Furthermore, one directly obtains $\langle \tilde{w}_n|1\rangle = 0$ from the orthogonality of the Fourier-Legendre remainder w_n with $l_0(\theta) = 1$ (see Lemma 1). Therefore, Wirtinger's first inequality (16) states that, under the equality and integral conditions, the inequality $\|\tilde{w}_n'\| \geq \pi\|\tilde{w}_n\|$ holds. It remains to compute $\|\tilde{w}_n'\|$ and $\|\tilde{w}_n\|$. On the first hand,

$$\begin{aligned}\|\tilde{w}_n'\|^2 &= \int_{-1}^{1} \left(w_n'(\theta) - l_{2n-1}'(\theta) w_n(1)\right)^2 \mathrm{d}\theta, \\ &= \|w_n'\|^2 - 2\left\langle l_{2n-1}' \middle| w_n'\right\rangle w_n(1) + \left\langle l_{2n-1}' \middle| l_{2n-1}'\right\rangle w_n^2(1).\end{aligned}$$

By integration by parts, we can write

$$\|\tilde{w}_n'\|^2 = \|w_n'\|^2 + 2\left(\left\langle l_{2n-1}'' \middle| w_n \right\rangle - [l_{2n-1}' w_n]_{-1}^{1}\right) w_n(1) + \left\langle l_{2n-1}' \middle| l_{2n-1}' \right\rangle w_n^2(1).$$

Using Lemma 1 verified by w_n, we can get rid of the second term and say that

$$\|\tilde{w}_n'\|^2 = \|w_n'\|^2 - \sum_{p=0}^{n-1} 2(4p+1) w_n^2(1),$$

The last term of the previous expression has already been computed in (15), and we know that

$$\|\tilde{w}_n'\|^2 = \|w_n'\|^2 - 2n(2n-1) w_n^2(1), \tag{19}$$

On the other hand, the norm of $\tilde{w}_n$ can be computed as follows. Due to the orthogonality the Legendre polynomials provided by Lemma 1, we have

$$\|\tilde{w}_n\|^2 = \|w_n\|^2 + \frac{2}{4n-1} w_n^2(1). \tag{20}$$

Thus, the proof is concluded by merging the two expressions given in (19) and (20) into $\|\tilde{w}_n'\| \geq \pi \|\tilde{w}_n\|$. ■

The previous lemma extends the Wirtinger inequality, in the situation where z is equal to 0 at $\theta = 0$. The main advantages of using the Fourier-Legendre remainder appears in the simple formulation of the lower bound in (17). It is important to stress that the orthogonality condition $\langle \ell_n | w_n \rangle = 0$ drastically simplifies the expression and the calculations.

It is worth mentioning that both constants κ_n^B and $\kappa_n^W = \kappa_{n-1}^B + \frac{2\pi^2}{4n-1}$ increase in a quadratic manner with respect to n. This information will not be used in this paper but could be a key point in the proof of the convergence of the sufficient stability criterion developed in the last section.

4 Modeling of an augmented system

In this section, one proposes the design of an augmented model issued from the interconnected system (1). The introduction of such model will allow to perform an assurate stability test.

4.1 Transfer function of the reaction-diffusion equation

Take the reaction-diffusion equation with cross type boundary conditions:

$$\begin{cases} \partial_t z(t,\theta) = (\delta\partial_{\theta\theta} + \lambda) z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_{\geq 0} \times (0,1), \quad (21a) \\ \left[\begin{smallmatrix} z(t,0) \\ \partial_\theta z(t,1) \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 \\ e(t) \end{smallmatrix} \right], & \forall t \in \mathbb{R}_{\geq 0}, \quad (21b) \end{cases}$$

where $e(t)$ is an input signal.

In the following, system (21) is transformed in Laplace domain where $s \in \mathbb{C}$ is the Laplace variable and where variables in capital letters represents the Laplace transform associated with variable in non-capital letters. As an infinite-dimensional system, the irrational transfer function between $E(s) = \frac{\partial}{\partial\theta} Z(s,1)$ and $Z(s,\theta)$ can be expressed as in [16] by

$$G(s,\theta) = \frac{\mathrm{sh}\left(\theta\sqrt{\frac{s-\lambda}{\delta}}\right)}{\sqrt{\frac{s-\lambda}{\delta}}\mathrm{ch}\left(\sqrt{\frac{s-\lambda}{\delta}}\right)}, \quad \forall (s,\theta) \in \mathbb{C} \times [-1,1]. \tag{22}$$

The main issue is to approximate this infinite-dimensional part in a specific manner and to highlight the link with Legendre approximation methods.
As $G(s,\theta)$ is an odd analytical function, it is possible to define Fourier-Legendre remainder $\tilde{G}_n$ done by truncation at order n of the Legendre-Fourier series of G on Legendre polynomials coefficients which is, for all $s \in \mathbb{C}$,

$$\tilde{G}_n(s,\theta) = G(s,\theta) - \sum_{k=0}^{n-1} \frac{4k+3}{2} l_{2k+1}(\theta) \langle l_{2k+1} | G(s) \rangle, \quad \forall \theta \in [-1,1]. \tag{23}$$

Remark 3 This remainder (23) is well-defined on the segment $[-1,1]$ with respect to θ by the fact that $G(s)$ belongs to $C^\infty([-1,1],\mathbb{C})$ for any s in $\mathbb{C}$ and $G_n(s)$ converges uniformly on any compact subset of $\mathbb{C}$ [27].

Then, error (23) can be rewritten as

$$\tilde{G}_n(s,\theta) = \sum_{k=n}^{\infty} \frac{4k+3}{2} l_{2k+1}(\theta) \int_{-1}^{1} G(s,\theta) l_{2k+1}(\theta) d\theta, \quad \forall \theta \in [-1,1], \tag{24}$$

and an estimation of its value around $s = \lambda$ can be proposed.

Lemma 6 *Error (23) is* $\underset{s\to\lambda}{O}(s^n)$*, for all* $\theta \in [-1,1]$.

Proof Lemma 6 means that $\tilde{G}_n(s,\theta)$ has its $n-1$ first derivatives with respect to s evaluated at $s = \lambda$ equal to zero and this is proven herafter. From the Maclaurin development of G around $s = \lambda$ with an infinite radius of convergence, for all $\theta \in [-1,1]$,

$$G(s,\theta) = \frac{1}{\mathrm{ch}\left(\sqrt{\frac{s-\lambda}{\delta}}\right)} \sum_{k=n}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!} \left(\frac{s-\lambda}{\delta}\right)^k .$$

The p successive derivations of G with respect to s evaluated at $s = \lambda$ give

$$\frac{\partial^p}{\partial s^p} G(\lambda, \theta) = \sum_{k=0}^{p} \frac{k! \theta^{2k+1}}{(2k+1)!}.$$

Regarding (24), one obtains zero for $p < n$ because θ^{2p} can be decomposed on the $2p + 1$ first Legendre polynomials. ■

This error is close to zero for s near λ and allows us to expect that it leads to accurate models, which satisfy the definition of the Padé approximants [3].

4.2 Realization of the $(n-1|n)$ Padé approximant of the transfer function taking support on the first Legendre polynomials

In this part, with the measurement $y(t) = z(t, 1)$, the irrational transfer function $H(s)$ of the reaction-diffusion equation from $E(s)$ to $Y(s)$

$$H(s) = G(s, 1) = \frac{\mathrm{th}\left(\sqrt{\frac{s-\lambda}{\delta}}\right)}{\sqrt{\frac{s-\lambda}{\delta}}}, \quad \forall s \in \mathbb{C}, \tag{25}$$

is considered. We show the relation between the modeling inspired by tau methods [13] on Legendre polynomials basis and the $(n-1|n)$ Padé approximated transfer function H_n of the irrational transfer function $H(s)$. Particular state form representation of this nice Padé approximated model is provided. Indeed, the states of this realization are Fourier-Legendre polynomials coefficients of the distributed state $z(t, \theta)$ under the following definition.

Definition 2 For any signal $z(t) \in \mathcal{L}^2(0, 1; \mathbb{R})$ satisfying (21) and for integer n in $\mathbb{N}$, we define,

- the state $z(t)$ on the extended interval $[-1, 1]$ by

$$z(t, \theta) = -z(t, -\theta), \quad \forall (t, \theta) \in \mathbb{R}_{\geq 0} \times [-1, 0], \tag{26}$$

- the n first Fourier-Legendre coefficients of z as

$$\zeta_n(t) = \begin{bmatrix} \int_{-1}^{1} l_1(\theta) z(t, \theta) \mathrm{d}\theta \\ \vdots \\ \int_{-1}^{1} l_{2n-1}(\theta) z(t, \theta) \mathrm{d}\theta \end{bmatrix} = \langle \ell_n | z(t) \rangle \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{27}$$

- the associated Fourier-Legendre remainder of $z(t)$ at order n as

$$w_n(t, \theta) = z(t, \theta) - \ell_n^\top(\theta) \mathcal{I}_n \zeta_n(t), \quad \forall (t, \theta) \in \mathbb{R}_{\geq 0} \times [-1, 1]. \tag{28}$$

Then, we build a finite-dimensional approximation of system (21) taking support on Legendre polynomials coefficients and certify that the approximated model matches with Padé approximants, thanks to Lemmas 6 and 7.

Proposition 3 *System* $\left(\begin{smallmatrix}\mathcal{A}_n & \mathcal{B}_n \\ C_n & 0\end{smallmatrix}\right)$ *is a realization of the* $(n-1|n)$ *Padé approximant of the transfer function* $H(s)$ *given by* (25) *with*

$$\begin{aligned}
&\mathcal{A}_n = \lambda I_n + \delta \mathcal{L}_n + \mathcal{B}_n^* C_n, \\
&\mathcal{B}_n = 2\delta \ell_n(1), \quad \mathcal{B}_n^* = -2\delta \ell_n'(1), \quad C_n = \ell_n(1)^T \mathcal{I}_n, \quad C_n^* = -\ell_n^{'\top}(1)\mathcal{I}_n,
\end{aligned} \tag{29}$$

where $\mathcal{L}_n, \mathcal{I}_n$ *are given in* (8) *and* $\ell_n(1), \ell_n'(1)$ *recalled in* (9).

Proof By two successive integrations by parts, the dynamics of the n coefficients $\zeta_n(t)$ defined by (27) lead to

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\zeta_n(t) &= \delta \left\langle \ell_n | \partial_{\theta\theta} z(t) \right\rangle + \lambda \zeta_n(t), \\
&= -\delta \left\langle \ell_n' \middle| \partial_\theta z(t) \right\rangle + \delta \left[\ell_n(\theta)\partial_\theta z(t,\theta)\right]_{-1}^{1} + \lambda \mathbf{z}_n(t), \\
&= \delta \left\langle \ell_n'' \middle| z(t) \right\rangle + \lambda \zeta_n(t) + \delta \left[\ell_n(\theta)\tfrac{\partial}{\partial\theta} z(t,\theta)\right]_{-1}^{1} - \delta \left[\ell_n'(\theta) z(t,\theta)\right]_{-1}^{1}, \\
&= (\lambda I_n + \delta \mathcal{L}_n)\zeta_n(t) + 2\delta \ell_n(1) e(t) - 2\delta \ell_n'(1) z(t,1).
\end{aligned}$$

The signal $y(t) = z(t,1)$ can be approximated by its the truncated Fourier-Legendre series at order n and called $w_n(t,1)$. That implies

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}\zeta_n(t) = \mathcal{A}_n \zeta_n(t) + \mathcal{B}_n e(t) + \mathcal{B}_n^* w_n(t,1), \\
\quad y(t) = z(t,1) = C_n \mathbf{z}_n(t) + w_n(t,1).
\end{cases} \tag{30}$$

In Laplace domain, the error $\mathcal{W}_n(s,1) = \tilde{H}_n(s) U(s)$ with

$$\tilde{H}_n(s) = \tilde{G}_n(s,1) = H(s) - C_n \left\langle \ell_n | G(s) \right\rangle. \tag{31}$$

It is the remainder of the truncated Fourier-Legendre series of G evaluated at $\theta = 1$. System (30) leads to

$$\begin{cases}
\mathcal{Z}_n(s) = (sI_n - \mathcal{A}_n)^{-1}(\mathcal{B}_n + \mathcal{B}_n^* \tilde{H}_n(s)) U(s), \\
\ Y(s) = C_n \mathcal{Z}_n(s) + \tilde{H}_n(s) U(s).
\end{cases}$$

The transfer function $H(s)$ from $E(s)$ to $Y(s)$ is

$$H(s) = C_n (sI_n - \mathcal{A}_n)^{-1}\mathcal{B}_n + \left(1 + C_n (sI_n - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right)\tilde{H}_n(s).$$

Thanks to Lemma 6, we already have $\tilde{H}_n(s) = \underset{s\to\lambda}{O}(s^n)$. Then, by application of Lemma 7 given in Appendix with vectors $u = -\mathcal{B}_n^*$, $v^T = C_n$ and matrix $L = -\delta \mathcal{L}_n$ which satisfy the expected assumptions, we find

$$H(s) - C_n (sI_n - \mathcal{A}_n)^{-1}\mathcal{B}_n = \underset{s\to\lambda}{O}(s^{2n}).$$

According to the definition of Padé approximations given in [3], the rational function $H_n(s) = C_n(sI_n - \mathcal{A}_n)^{-1}\mathcal{B}_n$ is a $(n-1|n)$ Padé approximant of $H(s)$ at $s = \lambda$. ∎

4.3 Modeling of the interconnected system

Consider now (x, z), solution to system (1) and, resuming Definition 2, denote the first Fourier-Legendre coefficients by ζ_n and the Fourier-Legendre remainder w_n, for any n in $\mathbb{N}$. The objective of this section is to rewrite system (1) by exhibiting a finite-dimensional part composed of the $\xi_n = \left[\begin{smallmatrix}\zeta_n \\ x\end{smallmatrix}\right]$ and an infinite-dimensional part represented by the Fourier-Legendre remainder w_n. This is formulated in the following proposition.

Proposition 4 *If (x, z) is a solution of system* (1)*, then $(\xi_n = \left[\begin{smallmatrix}\zeta_n \\ x\end{smallmatrix}\right], w_n)$ defined by* (27),(28) *verifies the following dynamics*

$$\begin{cases} \dot{\xi}_n(t) = \mathbf{A}_n\xi_n(t) + \mathbf{B}_n w_n(t,1), & \text{(32a)} \\ \partial_t w_n(t,\theta) = (\delta\partial_{\theta\theta} + \lambda)w_n(t,\theta) - \ell_n^\top(\theta)\mathcal{I}_n\big(\mathbf{E}_n\xi_n(t) + \mathbf{F}_n w_n(t,1)\big), & \text{(32b)} \\ \left[\begin{smallmatrix} w_n(t,0) \\ \partial_\theta w_n(t,1)\end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \mathbf{C}_n\xi_n(t)\end{smallmatrix}\right], & \text{(32c)} \\ (\xi_n(0), w_n(0)) = \left(\left[\begin{smallmatrix} x_0 \\ \zeta_{0,n}\end{smallmatrix}\right], w_{0,n}\right), & \text{(32d)} \end{cases}$$

where the matrices that defined this model are given by

$$\mathbf{A}_n = \begin{bmatrix} \mathcal{A}_n & \mathcal{B}_n C \\ BC_n & A \end{bmatrix},\ \mathbf{B}_n = \begin{bmatrix} \mathcal{B}_n^* \\ B \end{bmatrix},\ \mathbf{C}_n = \begin{bmatrix} C_n^* & C \end{bmatrix},\ \mathbf{E}_n = \mathcal{B}_n\mathbf{C}_n,\ \mathbf{F}_n = \mathcal{B}_n^*, \quad (33)$$

with matrices $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{B}_n^*$, C_n and C_n^* are given by (29)*.*

Proof The proof is also split into three parts referring to each equation in (32).
Proof of (32a): According to the proof Proposition 3, we already have $\zeta_n = \langle \ell_n | z\rangle$ which satisfies (30). Then, adding the dynamics of the ordinary differential equation given by (1a), the dynamics of the finite dimensional state are

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \zeta_n(t) \\ x(t) \end{bmatrix}}_{\frac{\mathrm{d}}{\mathrm{d}t}\xi_n(t)} = \underbrace{\begin{bmatrix} \mathcal{A}_n & \mathcal{B}_n C \\ BC_n & A \end{bmatrix}}_{\mathbf{A}_n}\underbrace{\begin{bmatrix} \zeta_n(t) \\ x(t) \end{bmatrix}}_{\xi_n(t)} + \underbrace{\begin{bmatrix} \mathcal{B}_n^* \\ B \end{bmatrix}}_{\mathbf{B}_n} w_n(t,1), \quad (34)$$

which corresponds to the first equation (32a).
Proof of (32b): Let us now focus in the second equation (32b), which refers to the partial differential equation resulting from the changes of variable z to w_n. To do so, differentiating with respect to time of the Fourier-Legendre remainder w_n given

in (28) yields $\partial_t w_n(t,\theta) = \partial_t z(t,\theta) - \ell_n^\top(\theta)\mathcal{I}_n \frac{\mathrm{d}}{\mathrm{d}t}\zeta_n(t)$. From one side, we need to express $\partial_t z$ using the new system of coordinates, that is reflected in

$$\begin{aligned}\partial_t z(t,\theta) &= \delta\partial_{\theta\theta} z(t,\theta) + \lambda z(t,\theta)\\ &= (\delta\partial_{\theta\theta} + \lambda) w_n(t,\theta) + \left(\delta\partial_{\theta\theta}\ell_n^\top(\theta) + \lambda\ell_n^\top(\theta)\right)\mathcal{I}_n\zeta_n(t).\end{aligned}$$

Applying the differentiation rules of the Legendre polynomials in (7b), the previous expression resumes to

$$\partial_t z(t,\theta) = (\delta\partial_{\theta\theta} + \lambda) w_n(t,\theta) + \ell_n^\top(\theta)\left(\delta\mathcal{L}_n^\top + \lambda I_n\right)\mathcal{I}_n\zeta_n(t). \tag{35}$$

On the other side, the expression of $\frac{\mathrm{d}}{\mathrm{d}t}\zeta_n(t)$ given by (30) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_n(t) = \mathcal{I}_n^{-1}(\delta\mathcal{L}_n^\top + \lambda I_n)\mathcal{I}_n\zeta_n(t) + \underbrace{\mathcal{B}_n\left[\begin{smallmatrix} C_n^* & C\end{smallmatrix}\right]}_{\mathbf{E}_n}\left[\begin{smallmatrix}\zeta_n(t)\\ x(t)\end{smallmatrix}\right] + \underbrace{\mathcal{B}_n^*}_{\mathbf{F}_n} w_n(t,1), \tag{36}$$

and calcultation details can be found in [2]. Thus, collecting (35),(36) and simplifying the term in $\delta\mathcal{L}_n^\top + \lambda I_n$, the partial differential equation verified by w_n is recognized. Proof of (32b): Finally, the first boundary condition is already verified according to Lemma 1 and the second boundary condition is directly obtained by derivation of (28) and evaluation at $\theta = 1$. ■

Remark 4 Matrix $\mathbf{A}_n$ given by (33) is equal to

$$\mathbf{A}_n = \left(\begin{smallmatrix}\mathcal{A}_n & \mathcal{B}_n\\ C_n & 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} A & B\\ C & 0\end{smallmatrix}\right) = H_n(s) \otimes \left(\begin{smallmatrix} A & B\\ C & 0\end{smallmatrix}\right), \tag{37}$$

where $\otimes$ denotes the Redheffer product and $H_n(s)$ the Padé $(n-1|n)$ approximant of $H(s)$ given by (25) at $s = \lambda$.

In the new formulation, the reaction-diffusion equation, which characterizes the dynamics of w_n, is similar to the one of the original system. The only difference relies on the last term in (32b). Even though, it seems at a first sight more complicated, it will appear in the next developments that this new term has no impact on the complexity of the analysis. This is due to the orthogonality of this new term with the Fourier-Legendre remainder, w_n.

5 Stability analysis

This last section is dedicated to the construction of a numerical tractable stability criterion for system (1), based on Lemmas 3 and 5 and highly related to the properties of the augmented model (32).

Theorem 1 *For a given integer n in $\mathbb{N}$ and any λ, δ satisfying $\frac{\lambda}{\delta} < \pi^2$, assume that there exists $\mathbf{P}_n$ in $\mathbb{S}_+^{n_x+n+1}$ such that matrix*

$$\Xi_n = \begin{cases} \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^\top \\ * & -\kappa_n^B \end{bmatrix}, & \textit{if } \lambda \leq 0, \\ \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^\top \\ * & -\left(1 - \frac{\lambda}{\delta\pi^2}\right)\kappa_n^B - \frac{\lambda}{\delta\pi^2}\kappa_n^W \end{bmatrix}, & \textit{if } \lambda > 0. \end{cases} \tag{38}$$

is negative definite, where constants κ_n^B*,* κ_n^W *are defined respectively by* (14),(17) *and matrices* $\mathbf{A}_n$*,* $\mathbf{B}_n$*,* $\mathbf{C}_n$ *and* $\mathbf{D}_n$ *are defined by* (33)*. Then, under the conditions prescribed by Propositions 1 and 2, the equilibrium* $(0,0)$ *of system* (1) *is globally exponentially stable, in the sense of the* $\mathbb{R}^{n_x} \times \mathcal{L}^2(0,1;\mathbb{R})$ *norm.*

Proof Consider the Lyapunov functional given by

$$\mathcal{V}_n(x,z) = \underbrace{\xi_n^\top \mathbf{P}_n \xi_n}_{\mathcal{V}_n^a(x,z)} + \underbrace{(2\delta)^{-1}\|w_n\|^2}_{\mathcal{V}_n^b(x,z)}, \tag{39}$$

with $\mathbf{P}_n$ in $\mathbb{S}_+^{n_x+n+1}$. It is a Lyapunov candidate functional for both systems (1),(32). Differentiation of $V_n^a(t) = \mathcal{V}_n^a(x,z)(t)$ along the trajectories of the system (1) using the dynamics given by (32) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n^a(t) = \xi_n^\top(t)\mathcal{H}(\mathbf{P}_n\mathbf{A}_n)\xi_n(t) + 2\xi_n^\top(t)\mathbf{P}_n\mathbf{B}_n w_n(t,1).$$

Secondly, from the dynamics of the Fourier-Legendre remainder in (32), denoting $V_n^b(t) = \mathcal{V}_n^b(x,z)(t)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n^b(t) = \int_{-1}^{1} \partial_{\theta\theta} w_n(t,\theta) w_n(t,\theta)\mathrm{d}\theta + \frac{\lambda}{\delta}\int_{-1}^{1} w_n^2(t,\theta)\mathrm{d}\theta.$$

Using integration by parts, this expression is decomposed in $\mathcal{L}^2(-1,1;\mathbb{R})$ norms of signals w_n and $\partial_\theta w_n$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_n^b(t) &= \frac{\lambda}{\delta}\|w_n(t)\|^2 - \|\partial_\theta w_n(t)\|^2 + 2\partial_\theta w_n(t,1) w_n(t,1), \\ &= \frac{\lambda}{\delta}\|w_n(t)\|^2 - \|\partial_\theta w_n(t)\|^2 + 2\xi_n^\top(t)\mathbf{C}_n^\top w_n(t,1).\end{aligned}$$

Thereafter, the proof is split into two cases. If $\lambda \leq 0$, the proof is a straightforward application of Bessel-Legendre inequality given by (14). Indeed, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) \leq -\frac{|\lambda|}{\delta}\|w_n(t)\|^2 + \begin{bmatrix} \xi_n(t) \\ w_n(t,1) \end{bmatrix}^\top \Xi_n \begin{bmatrix} \xi_n(t) \\ w_n(t,1) \end{bmatrix},$$

Assuming $\Xi_n \prec 0$, application of Lyapunov theorem leads to Theorem 1.
For the case $0 < \lambda < \delta\pi^2$, we apply first the adapted Wirtinger inequality (17) on the Fourier-Legendre remainder w_n to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_n^b(t) \leq &-\epsilon\pi^2\|w_n(t)\|^2 - \left(1-\frac{\lambda}{\delta\pi^2}-\epsilon\right)\|\partial_\theta w_n(t)\|^2 \\ &+2\xi_n^\top(t)\mathbf{C}_n^\top w_n(t,1) - \left(\frac{\lambda}{\delta\pi^2}+\epsilon\right)\kappa_n^W w_n^2(t,1),\end{aligned}$$

for any sufficiently small $\epsilon > 0$ such that $(1-\frac{\lambda}{\delta\pi^2}-\epsilon) > 0$. By application of the Bessel-Legendre inequality given by (14), we finally get to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_n^b(t) \leq &-\epsilon\pi^2\|w_n\|^2 + 2\xi_n^\top(t)\mathbf{C}_n^\top w_n(t,1) \\ &-\left((1-\frac{\lambda}{\delta\pi^2}-\epsilon)\kappa_n^B + (\frac{\lambda}{\delta\pi^2}+\epsilon)\kappa_n^W\right)w_n^2(t,1).\end{aligned}$$

Merged with $\frac{\mathrm{d}}{\mathrm{d}t}V_n^a(t)$, it gives

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) \leq -\epsilon\pi^2\|w_n(t)\|^2 + \begin{bmatrix}\xi_n(t)\\ w_n(t,1)\end{bmatrix}^\top \Xi_n \begin{bmatrix}\xi_n(t)\\ w_n(t,1)\end{bmatrix} + \epsilon\kappa_n^B w_n^2(t,1).$$

If the linear matrix inequality $\Xi_n \prec 0$ holds, it is possible to take ϵ small enough such that $-|\bar{\sigma}(\Xi_n)| + \epsilon\kappa_n^B < 0$. Then, there exists $\rho > 0$ such that the derivatives $\frac{\mathrm{d}}{\mathrm{d}t}V_n(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) \leq -\rho V_n(t)$. One concludes by application of Lyapunov theorem on the exponential stability at the equilibrium point. ■

Remark 5 Notice that $\mathbf{A}_n$ Hurwitz is a necessary condition for the feasibility of linear matrix inequality $\Xi_n \prec 0$ where Ξ_n defined by (38). This is promising to the matter of fact that $\mathbf{A}_n$ is an approximated model for the original system (1).

Remark 6 Compared to generalized linear matrix inequality formulations based on sum of squares [9], the result is condensed, more appropriate to the application and numerical burden are improved. This optimization is due to the transformations made to obtain our linear matrix inequality, highly correlated to the system under study. Nevertheless, to the best of our knowledge, we are not aware of stability condition adressing this particular class of system and this is the reason why no further comparison will be presented.

6 Application to numerical examples

6.1 On Padé approximations

Without loss of generalization, let take $\lambda = 0$ and $\delta = 1$. The approximated model $\left(\begin{smallmatrix}\mathcal{A}_n & \mathcal{B}_n\\ \mathcal{C}_n & 0\end{smallmatrix}\right)$ leads to a transfer function H_n. For low orders, the corresponding matrices and approximated transfer functions have been stored in Table 1.

Order n	1	2	3	4
$\mathcal{A}_n$	-3	$\begin{bmatrix} -3 & -7 \\ -3 & -42 \end{bmatrix}$	$\begin{bmatrix} -3 & -7 & -11 \\ -3 & -42 & -66 \\ -3 & -42 & -165 \end{bmatrix}$	$\begin{bmatrix} -3 & -7 & -11 & -15 \\ -3 & -42 & -66 & -90 \\ -3 & -42 & -165 & -225 \\ -3 & -42 & -165 & -420 \end{bmatrix}$
$\mathcal{B}_n$	2	$\begin{bmatrix} 2 & 2 \end{bmatrix}^\top$	$\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}^\top$	$\begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}^\top$
$\mathcal{C}_n$	$\frac{3}{2}$	$\begin{bmatrix} \frac{3}{2} & \frac{7}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{3}{2} & \frac{7}{2} & \frac{11}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{3}{2} & \frac{7}{2} & \frac{11}{2} & \frac{15}{2} \end{bmatrix}$
$H_n(s)$	$\frac{3}{s+3}$	$5\frac{2s+21}{s^2+45s+105}$	$21\frac{s^2+60s+495}{s^3+210s^2+4725s+10395}$	$9\frac{4s^3+770s^2+30030s+225225}{s^4+630s^3+51975s+2027025}$

Table 1 Padé $(n-1|n)$ approximants of transfer function $H(s)$.

Order n	1	2	3	4	Expected
$B = 2$	-3.751	-3.668	-3.668	-3.668	-3.668
$B = 0$	2.461	2.467	2.467	2.467	$(\frac{\pi}{2})^2 = 2.467$
$B = 0.1$	2.258	2.263	2.263	2.263	2.263
$B = -1$	3.176	3.672	3.685	3.685	3.685

Table 2 Maximal allowable reaction coefficient λ_{max} guaranteed by Theorem 1.

By comparison with the Padé approximant of $H(s) = G(s,1) = \frac{\text{sh}(\sqrt{s})}{\sqrt{s}\text{ch}(\sqrt{s})}$ at $s = 0$ computed by Matlab *pade* tool, the same transfer function H_n are recognized.
Then, according to Remark 4, the approximated model $\mathbf{A}_n$ will be equivalent to the star product of the finite-dimensional model with this Padé $(n-1|n)$ transfer function of the reaction-diffusion equation.

6.2 On stability results

In this subsection, a standard example is chosen. The stability properties with respect to λ are investigated with the method proposed in this chapter.

Example 1 Let system (1) with $A = -1$, B taking values in $[-1, 2]$, $C = 1$ and $\delta = 1$.

The feasibility of linear matrix inequality $\Xi_n \prec 0$ with Ξ_n given by (38) is determined with feasp function and tested for $n = 2$. The maximal bound λ_{max} which ensures the stability of the system is collected in Table 2 for several values of B. One compares this result with expected bounds coming from the stability of an approximated model at order 100 (see tau methods [12]).
According to Table 2, we can see that an upper bound for parameter λ has been found to guarantee exponential stability of the origin of system (1). Based on cases $B < 0$, it is important to note that the classical upper bound $(\frac{\pi}{2})^2$ can be exceeded thanks to the stabilizable properties of $A < 0$. It is also worth noticing that the proposed test seem to be hierarchical and to converge toward the expected bound of the region of stability in a fast manner (from order $n \simeq 3$). At last, the required order n seems to be larger as λ_{max} increases (i.e. B decreases).

7 Conclusions

From a Lyapunov approach, a new sufficient condition of stability for a system coupled with a reaction diffusion process is presented in the linear matrix framework. Our work focused on the rewriting of Bessel and Wirtinger inequalities involved in the Lyapunov analysis of the reaction-diffusion equation and signals linked with Legendre remainder emerged. Then, a dynamical model has been constructed to highlight these signals and the extended state composed of the first Legendre polynomials coefficients. Finally, taking a classical quadratic Lyapunov functional fitted to this new modeling, a linear matrix inequality is directly obtained.
Based on the structure of the result, robust approach could be pursued to construct finite-dimensional controllers stabilizing the original infinite-dimensional system. Observer design by early-lumping might also be investigated.

Appendix

Derived from the matrix inversion and determinant lemmas, a usefull lemma can also be formulated.

Lemma 7 *For any $u \in \mathbb{R}^n$ with a non-zero first component, $v \in \mathbb{R}^n$ not equal to the zero vector and $L \in \mathbb{R}^{n\times n}$ a strictly lower triangular matrix with non-zero under diagonal coefficients (i.e. $L_{i+1,i} \neq 0 \; \forall i \in \{1, \dots n-1\}$), one obtains*

$$1 - v^T((s-\lambda)I_n + L + uv^T)^{-1}u = \underset{s\to\lambda}{O}(s^n). \tag{40}$$

Proof The matrix inversion lemma applied to vectors $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$ and non singular matrix $M = sI_n + L \in \mathbb{R}^{n\times n}$ gives, for any $s \in \mathbb{C}\backslash\{0\}$,

$$1 - v^T((s-\lambda)I_n + L + uv^T)^{-1}u = (1 + v^T((s-\lambda)I_n + L)^{-1}u)^{-1},$$

and the matrix determinant lemma leads to

$$1 - v^T((s-\lambda)I_n + L + uv^T)^{-1}u = \frac{\det((s-\lambda)I_n + L)}{\det((s-\lambda)I_n + L + uv^T)}.$$

Then, since L is strictly lower triangular, we have

$$\det((s-\lambda)I_n + L) = \det((s-\lambda)I_n) = (s-\lambda)^n.$$

and, because L has non-zero under diagonal coefficients and under the hypothesis done on vectors u, v, matrix $L + uv^T$ has full rank which means $\det(L + uv^T) \neq 0$. That yields the result for s tending to λ. ■

References

1. Aarsnes, U., Vazquez, R., Di Meglio, F., Krstic, M.: Delay robust control design of under-actuated PDE-ODE-PDE systems. In: 2019 American Control Conference (ACC), pp. 3200–3205. IEEE (2019)
2. Bajodek, M., Seuret, A., Gouaisbaut, F.: Stability analysis of an ordinary differential equation interconnected with the reaction-diffusion equation. Automatica (2021, under review)
3. Baker, G., Graves-Morris, P.: Padé Approximants, 2 edn. Encyclopedia of Mathematics and its Applications $n°59$ (1996)
4. Barreau, M., Gouaisbaut, F., Seuret, A., Sipahi, R.: Input/output stability of a damped string equation coupled with ordinary differential system. International Journal of Robust and Non-linear Control **28**(18), 6053–6069 (2018)
5. Baudouin, L., Seuret, A., Gouaisbaut, F.: Stability analysis of a system coupled to a heat equation. Automatica pp. 195–202 (2019)
6. Curtain, R., Morris, K.: Transfer functions of distributed parameter systems: A tutorial. Automatica **45**(5), 1101–1116 (2009)
7. Curtain, R., Zwart, H.: An Introduction to Infinite-Dimensional Linear Systems Theory. Springer (1995)
8. Das, A., Shivakumar, S., Peet, M., Weiland, S.: Robust analysis of uncertain ODE-PDE systems using PI multipliers, PIEs and LPIs. In: 2020 59th IEEE Conference on Decision and Control (CDC), pp. 634–639. IEEE (2020)
9. Das, A., Shivakumar, S., Weiland, S., Peet, M.: Representation and stability analysis of PDE-ODE coupled systems (2018). ArXiv:1812.07186
10. Evans, L.: Partial Differential Equations. American Mathematical Society (1997)
11. Gautschi, W.: Orthogonal polynomials, quadrature, and approximation: computational methods and software (in Matlab). Lecture Notes in Mathematics **1883**, 1–77 (2006)
12. Gottlieb, D., Orszag, S.: Numerical analysis of spectral methods. Society for Industrial and Applied Mathematics, Philadelphia (1977)
13. Ito, K., Teglas, R.: Legendre-Tau approximations for functional differential equations. SIAM Journal on Control and Optimization **24**(4), 737–759 (1986)
14. Kammler, W.: A first course in Fourier analysis. Cambridge University Press (2007)
15. Karafyllis, I., Krstic, M.: Input-to-state stability for PDEs. Springer (2018)
16. Krstic, M.: Delay compensation for nonlinear, adaptive, and PDE systems. Springer (2009)
17. Lasieka, I., Triggiani, R.: Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Cambridge University Press (2000)
18. Lhachemi, H., Shorten, R.: Boundary feedback stabilization of a reaction–diffusion equation with Robin boundary conditions and state-delay. Automatica **116**, 108931 (2020)
19. Logemann, H.: Circle criteria, small-gain conditions and internal stability for infinite-dimensional systems. Automatica **27**(4), 677–690 (1991)
20. Mironchenko, A., Prieur, C.: Input-To-State Stability of Infinite-Dimensional Systems: Recent Results and Open Questions (2019)
21. Prieur, C., Girard, A., Witrant, E.: Stability of switched linear hyperbolic systems by Lyapunov techniques. IEEE Transactions on Automatic Control (2014)
22. Prieur, C., Trelat, E.: Feedback Stabilization of a 1-D Linear Reaction-Diffusion Equation with Delay Boundary Control. IEEE Transactions on Automatic Control **64**(4), 1415–1425 (2019)
23. Safi, M., Baudouin, L., Seuret, A.: Tractable sufficient stability conditions for a system coupling linear transport and differential equations. Systems and Control Letters **110**, 1–8 (2017)
24. Selivanov, A., Fridman, F.: Boundary observers for a reaction-diffusion system under time-delayed and sampled-data measurements. IEEE Transactions on Automatic Control **64**(8), 3385–3390 (2019)
25. Valmorbida, G., Ahmadi, M., Papachristodoulou, A.: Stability Analysis for a Class of Partial Differential Equations via Semidefinite Programming. IEEE Transactions on Automatic Control **61**(6), 1649–1654 (2016)

26. Wang, Z., Zhang, H., Li, P.: An LMI approach to stability analysis of reaction-diffusion Cohen-Grossberg neural networks concerning Dirichlet boundary conditions and distributed delays. IEEE Transactions on Systems, Man, and Cybernetics **40**(6), 1596–1606 (2010)
27. Webb, M., Coppé, V., Huybrechs, D.: Pointwise and uniform convergence of Fourier extensions. Constructive Approximation **52**, 139–175 (2019)