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CONVEXLY INDEPENDENT SUBSETS OF MINKOWSKI SUMS OF CONVEX POLYGONS

MATEUSZ SKOMRA\textsuperscript{1} AND STÉPHAN THOMASSÉ\textsuperscript{2,3}

Abstract. We show that there exist convex \(n\)-gons \(P\) and \(Q\) such that the largest convex polygon in the Minkowski sum \(P + Q\) has size \(\Theta(n \log n)\). This matches an upper bound of Tiwary.

1. Introduction

Let \(X\) be a finite set of points in the plane. A subset \(C\) of \(X\) is convexly independent if \(C\) forms a convex polygon. We denote by \(\text{ci}(X)\) the largest size of a convexly independent subset of \(X\). The celebrated Happy Ending Theorem, from Erdős and Szekeres \cite{erdos1935geometric}, asserts that \(\text{ci}(X)\) goes to infinity when \(|X|\) goes to infinity and \(X\) does not have three points on the same line. More precisely, the minimum value one can achieve for \(\text{ci}(X)\) is logarithmic in \(|X|\). To the opposite, one can try to maximize \(\text{ci}(X)\) when the set \(X\) satisfies some geometrical constraint. A well-studied case is when \(X\) is the Minkowski sum \(P + Q := \{p + q : p \in P, q \in Q\}\) of two sets of points \(P\) and \(Q\).

Eisenbrand, Pach, Rothvoß, and Sopher \cite{eisenbrand2008maximum} proved that \(\text{ci}(P + Q) = O(n^{4/3})\) when \(|P| = |Q| = n\). This result was complemented by a construction of Bílka, Buchin, Fulek, Kiyomi, Okamoto, Tanigawa, and Tóth \cite{bilkakiyomi2017maximal} showing the existence of such \(P\) and \(Q\) satisfying \(\text{ci}(P + Q) = \Theta(n^{4/3})\). Surprisingly, the set \(Q\) they use in the extremal constructions can be chosen convex. A natural question is then to ask for the maximum possible value of \(\text{ci}(P + Q)\) when both \(P\) and \(Q\) are convex polygons. In 2014, Tiwary \cite{tiwary2014maximum} proposed an upper bound by showing that \(\text{ci}(P + Q) = O((n + m) \log(n + m))\) when \(P\) and \(Q\) are respectively a convex \(n\)-gon and a convex \(m\)-gon. He concluded his paper by mentioning that his upper bound seemed very generous, and left as an open problem the existence of a matching lower bound. Our main result in this paper is that Tiwary’s proof indeed provides a sharp bound by exhibiting a matching construction.

**Theorem 1.1.** There exist two families \((P_k)_{k \geq 1}, (Q_k)_{k \geq 1} \subset \mathbb{R}^2\) of convexly independent sets such that \(|P_k| = |Q_k| = 2^k\) and \(\text{ci}(P_k + Q_k) \geq (k + 2)2^{k-1}\) for all \(k \geq 1\).

An equivalent point of view of Minkowski sums is to consider \((P + Q)/2\) instead of \(P + Q\), and therefore \(\text{ci}(P + Q)\) represents the maximum size of a set \(M\) of midpoints of \(P, Q\)-segments in convex position. If we actually draw

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all segments corresponding to these midpoints, we get a bipartite graph with vertex set $P \cup Q$, whose edges are all pairs $(p, q) \in P \times Q$ such that $(p+q)/2 \in M$. In our construction, the number of edges of this graph is $cn \log n$, all vertices of $P$ and $Q$ are in convex position, and all midpoints of edges are also in convex position. This kind of drawing was introduced by Halman, Onn, and Rothblum [8] where they define a strong convex embedding of a graph $G = (V, E)$ as a function $f: V \to \mathbb{R}^2$ such that $f(V)$ is convex and $\{(f(x) + f(y))/2: (x, y) \in E\}$ is also convex. They showed that if $G$ admits a strong convex embedding, then $|E| \leq 5n - 8$ when $n \geq 3$ is the number of vertices. Recently, García-Marco and Knauer [7] reduced this bound to $2n - 3$. Equivalently their result shows that $\text{ci}(P + P) \leq 2n - 3$ when $P$ is a convex $n$-gon. Perhaps surprisingly, our construction shows that when slightly relaxing strong convex embedding to only ask convex positions for the two partite sets of a bipartite graph, the bound goes from linear to $n \log n$.

Another very attractive reason motivating the study of $\text{ci}(P + Q)$ for convex $n$-gons is the famous unit distance problem of Erdős and Moser [4] asking for the maximum number of pairs of points in a convex $n$-gon $P$ with distance exactly one. The trick is to observe that if two points $p, p'$ of $P$ have distance 1, then $p - p'$ lies on the unit circle. A careful counting shows, in particular, that $\text{ci}(P + (-P))$ is an upper bound on the number of unit distance pairs. Unfortunately, our construction shows that $\text{ci}(P + Q)$ cannot be directly used to improve the already known $O(n \log n)$ upper bound on the unit distance problem (see Füredi [6] and Aggarwal [1] for a sharper estimate). We do not know however if the graphs $G_k$ arising from our construction can be realized as convex unit distance graphs. If not, it would be interesting to find a forbidden pattern in $G_k$. This would extend the list of forbidden matrices provided in [1], as none of them appears in $G_k$.

2. Proof of the main theorem

The proof of Tiwary [9] is based on the fact that any collection of convexly independent points in $\mathbb{R}^2$ can be split into at most four monotone chains as in Figure 1. We use the name “south-east chain” for the chain that is situated in the bottom-right part of this figure. Our construction is based on this type of chains. More formally, we make the following definition and point out the subsequent lemma.

Definition 2.1. A south-east chain is a sequence $(a^{(1)}, \ldots, a^{(n)}) \subset \mathbb{R}^2$ of $n \geq 2$ points in the plane that satisfies the following two conditions. First, the sequence is strictly increasing on both coordinates, i.e., we have $a^{(1)}_1 < a^{(2)}_1 < \cdots < a^{(n)}_1$ and $a^{(1)}_2 < a^{(2)}_2 < \cdots < a^{(n)}_2$ (where $a^{(k)}_i$ denotes the $i$th coordinate of the point $a^{(k)}$). Second, the corresponding sequence of consecutive slopes is strictly increasing, i.e., we have

$$\frac{a^{(2)}_2 - a^{(1)}_2}{a^{(2)}_1 - a^{(1)}_1} < \cdots < \frac{a^{(k+1)}_2 - a^{(k)}_2}{a^{(k+1)}_1 - a^{(k)}_1} < \cdots < \frac{a^{(n)}_2 - a^{(n-1)}_2}{a^{(n)}_1 - a^{(n-1)}_1}. $$

We say that $n$ is the length of a south-east chain $(a^{(1)}, \ldots, a^{(n)})$. 

 Lemma 2.2. If the points \((a^{(1)},\ldots,a^{(n)})\) form a south-east chain, then they are convexly independent, i.e., the convex hull of \(\{a^{(1)},\ldots,a^{(n)}\}\) has \(n\) vertices.

Before giving a formal proof of Theorem 1.1 let us explain it intuitively. Our proof is based on some elementary properties of rotations. Let \(R: \mathbb{R}^2 \to \mathbb{R}^2\) denote the counterclockwise rotation by 60 degrees centered at zero. Let \(a,b,c \in \mathbb{R}^2\) be three points on a plane such that \((a,b,c)\) is a south-east chain. Moreover, suppose that this chain is sufficiently flat (more precisely, that the line defined by extending the segment \([b,c]\) forms an angle smaller than 30 degrees with the horizontal axis). Then, the images \((R(a),R(b),R(c))\) also form a south-east chain. Moreover, if all the slopes between consecutive points of \((a,b,c)\) are close to 0, then the corresponding slopes of \((R(a),R(b),R(c))\) are close to \(\sqrt{3}\) (in other words, the corresponding segments form angles close to 60 degrees with the horizontal axis). Even more, under these conditions, the triple \(\frac{1}{2}(a + R(a),b + R(b),c + R(c))\) also forms a south-east chain, and its slopes are close to \(\frac{1}{\sqrt{3}}\) (which corresponds to the angle of 30 degrees). We refer to Figure 2 for an illustration. The same applies to south-east chains formed by more than three points. In
this way, given a sufficiently flat south-east chain, we can construct two new chains, one with slopes close to $\sqrt{3}$ and one with slopes close to $1/\sqrt{3}$.

Consider now the linear map $L_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $L_\varepsilon(x, y) := (\varepsilon x, \varepsilon^2 y)$, where $\varepsilon > 0$. It is immediate to see that this map preserves south-east chains (i.e., an image of a south-east chain under $L_\varepsilon$ is again a south-east chain). Moreover, this map flattens the chains. In other words, if $A$ is a south-east chain and $\varepsilon > 0$ is small enough, then the image of $A$ under $L_\varepsilon$ is a south-east chain that is sufficiently flat to apply the previous observations. Moreover, for small $\varepsilon$, the image of $A$ under $L_\varepsilon$ is contained in a small neighborhood of $0$.

Suppose now that we are given three south-east chains $A, B, C$ such that $C$ is included in the set of points $(A + B)/2$. By applying $L_\varepsilon$ to all three chains, we can suppose that they are arbitrarily flat, and contained in a small neighborhood of $0$. Then, we can apply the rotation $R$ to $A, B, C$. In this way, we obtain three chains $A', B', C'$ that are again contained in a neighborhood of $0$, but whose slopes are close to $\sqrt{3}$. We now translate the chains as follows. We do not apply any translation to $A$, we translate $B$ by the vector $(0, 2)$, and $C$ by the vector $(0, 1)$. Then, we translate $A'$ by the vector $(1, 5/2)$, $B'$ by the vector $(1, 1)$, and $C'$ by the vector $(1, 7/4)$. This gives the situation depicted in Figure 3. In this picture, the chain $A$ is contained in the small neighborhood of the point marked by $A$, and the slopes of this chain are close to the slope of the solid line passing through $A$. The same is true for the chains $B, C, A', B', C'$. In particular, for sufficiently
small \( \varepsilon \), the concatenation \( \bar{A} := (A,B') \) of chains \( A \) and \( B' \) forms a south-east chain (because the dashed line from \( A \) to \( B' \) has slope greater than the slope of the solid line passing through \( A \) but smaller than the slope of the solid line passing through \( B' \)). By the same reasoning, the concatenation \( \bar{B} := (B,A') \) forms a south-east chain. Denote \( A = (a^{(1)}, \ldots, a^{(n)}) \) and \( A' = (a'^{(1)}, \ldots, a'^{(n)}) \). By the observation about rotation made above, the sequence \( D := (\frac{a^{(1)} + a'^{(1)}}{2}, \ldots, \frac{a^{(n)} + a'^{(n)}}{2}) \) is a south-east chain, and all the slopes of this chain are close to \( \sqrt{3}/2 \). Moreover, this chain is contained in a small neighborhood of the point \((1/2, 5/4)\). We marked this chain in Figure 3 using the same conventions as for the remaining chains. By applying the same reasoning as above, the concatenation \( \bar{C} := (C,D,C') \) is a south-east chain. To summarize, our construction shows the following statement. Given three south-east chains \( A,B,C \) such that \( C \) is included in the set of points \((A + B)/2\), we can construct three south-east chains \( \bar{A}, \bar{B}, \bar{C} \) such that \( \bar{C} \) is included in \((\bar{A} + \bar{B})/2\) and \(|\bar{A}| = |\bar{B}| = |A| + |B|\), \(|\bar{C}| = 2|C| + |A|\). Thus, if we suppose that \(|A| = |B| = n\), then we have \(|A| = |\bar{B}| = 2n\) and \(|\bar{C}| = 2|C| + n\). By iterating this reasoning, we obtain the claimed bound \( \Theta(n \log n) \).

Before presenting a formal proof, let us discuss the types of graph drawings that we obtain in this way. Here, we are interested in a drawing \( f : (U \sqcup V) \to \mathbb{R}^2 \) of a bipartite graph \( G = (U \sqcup V, E) \) such that \( f(U) \) is a south-east chain, \( f(V) \) is a south-east chain, and the midpoints \( \{ (f(u) + f(v))/2 : (u,v) \in E \} \) also form a south-east chain. The construction described above implies that if \( G \) is drawable in this way and \( G' := (U' \sqcup V', E') \) is a copy of \( G \), then the graph \( G' := (U' \sqcup V, E) \) defined as \( U' := U \sqcup V' \), \( V' := V \sqcup U' \), and

\[
E := E \sqcup E' \sqcup \{ (u, u') : u \in U \}
\]

is also drawable in this fashion. By starting from a graph

\[
G_1 = (\{ u_1, u_2 \} \sqcup \{ v_1, v_2 \}, \{(u_1, v_1), (u_2, v_1), (u_2, v_2)\})
\]

and iterating the procedure, we obtain a family \( (G_k)_{k \geq 1} \) of drawable graphs, each having \( 2^{k+1} \) vertices and \( (k+2)2^{k-1} \) edges. Figure 4 depicts this family of graphs for \( k \leq 3 \) and Figure 5 depicts a drawing of \( G_3 \) using south-east chains.

In the remaining part of this section, we give a formal proof of the argument described above. Let \( R : \mathbb{R}^2 \to \mathbb{R}^2 \) denote the counterclockwise rotation by 60 degrees centered at zero, i.e., the linear transformation given
by the matrix
\[ R := \frac{1}{2} \begin{bmatrix} 1 & \frac{-\sqrt{3}}{} \\ \frac{\sqrt{3}}{} & 1 \end{bmatrix} \]

Furthermore, if \( a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2 \) are two points such that \( a_1 < b_1 \) and \( a_2 < b_2 \), then we denote by \( \text{sl}(a, b) := \frac{b_2 - a_2}{b_1 - a_1} \) the corresponding slope of the segment \([a, b] \).

The following lemma, which can be proven using elementary trigonometric identities, gathers the properties of rotation that were mentioned above.

**Lemma 2.3.** Suppose that two points \( a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2 \) are such that \( a_1 < b_1 \), \( a_2 < b_2 \), and \( \text{sl}(a, b) < \frac{1}{\sqrt{3}} = \tan \left( \frac{\pi}{6} \right) \). Let \( \theta := \arctan(\text{sl}(a, b)) < \frac{\pi}{6} \) and denote \( \tilde{a} = R(a) \), \( \tilde{b} = R(b) \). Then, we have \( \tilde{a}_1 < \tilde{b}_1 \), \( \tilde{a}_2 < \tilde{b}_2 \),

\[ \text{sl}(\tilde{a}, \tilde{b}) = \tan \left( \frac{\pi}{3} + \theta \right), \quad \text{and} \quad \text{sl} \left( \frac{a + \tilde{a}}{2}, \frac{b + \tilde{b}}{2} \right) = \tan \left( \frac{\pi}{6} + \theta \right). \]

**Proof.** We have \( \tilde{a} = \frac{1}{2}(a_1 - \sqrt{3}a_2, \sqrt{3}a_1 + a_2) \) and \( \tilde{b} = \frac{1}{2}(b_1 - \sqrt{3}b_2, \sqrt{3}b_1 + b_2) \).

The inequality \( \tilde{a}_2 < \tilde{b}_2 \) is trivial. Moreover, \( \tilde{b}_1 > \tilde{a}_1 \iff b_1 - a_1 > \sqrt{3}(b_2 - a_2) \), which is true by our assumptions. Furthermore, we have

\[ \tan \left( \frac{\pi}{3} + \theta \right) = \frac{\tan \left( \frac{\pi}{3} \right) + \tan(\theta)}{1 - \tan \left( \frac{\pi}{3} \right) \tan(\theta)} = \frac{\sqrt{3}(b_1 - a_1) + b_2 - a_2}{b_1 - a_1 - \sqrt{3}(b_2 - a_2)} = \frac{\tilde{b}_2 - \tilde{a}_2}{\tilde{b}_1 - \tilde{a}_1} = \text{sl}(\tilde{a}, \tilde{b}). \]
Similarly,
\[\tan\left(\frac{\pi}{6} + \theta\right) = \frac{\tan\left(\frac{\pi}{6}\right) + \tan(\theta)}{1 - \tan\left(\frac{\pi}{6}\right)\tan(\theta)} = \frac{b_1 - a_1 + \sqrt{3}(b_2 - a_2)}{\sqrt{3}(b_1 - a_1) - (b_2 - a_2)} = \frac{b_2 + \tilde{b} - a_2 - \tilde{a}}{b_1 + b_1 - a_1 - \tilde{a}} = \text{sl}\left(\frac{a + \tilde{a}}{2}, \frac{b + \tilde{b}}{2}\right).\]

For any \(\varepsilon > 0\) we denote by \(L_\varepsilon: \mathbb{R}^2 \to \mathbb{R}^2\) the linear transformation \(L_\varepsilon(x,y) := (\varepsilon x, \varepsilon^2 y)\). As a corollary of Lemma 2.3 we may now prove the properties of the three transformations of south-east chains discussed above. To improve readability, we use the following notation: if \(A\) is a sequence, then we denote by \(A(k)\) its \(k\)th element, so that \(A = (A(1), \ldots, A(n))\). If \(A\) is a south-east chain of length \(n\) and \(\varepsilon > 0\), then we consider the following three sequences:

\[
\begin{align*}
A_\varepsilon &:= \left(L_\varepsilon(A(1)), \ldots, L_\varepsilon(A(n))\right), \\
A'_\varepsilon &:= \left(R(A_\varepsilon(1)), \ldots, R(A_\varepsilon(n))\right), \\
A''_\varepsilon &:= \left(\frac{A_\varepsilon(1) + A'_\varepsilon(1)}{2}, \ldots, \frac{A_\varepsilon(n) + A'_\varepsilon(n)}{2}\right).
\end{align*}
\]

Using this notation, \(A_\varepsilon\) is a chain obtained by flattening \(A\), \(A'_\varepsilon\) is the rotated version of this flattened chain, and \(A''_\varepsilon\) is the chain formed by taking the midpoints of the two previous chains. The next result follows from Lemma 2.3.

**Lemma 2.4.** Suppose that \(A := (A(1), \ldots, A(n))\) is a south-east chain. Then, for sufficiently small \(\varepsilon > 0\), the sequences \(A_\varepsilon, A'_\varepsilon, A''_\varepsilon\) are south-east chains. Moreover, for every \(k \in [n-1]\) we have the equalities

\[
\begin{align*}
\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(k), A_\varepsilon(k+1)) &= 0, \\
\lim_{\varepsilon \to 0^+} \text{sl}(A'_\varepsilon(k), A'_\varepsilon(k+1)) &= \tan\left(\frac{\pi}{3}\right) = \sqrt{3}, \\
\lim_{\varepsilon \to 0^+} \text{sl}(A''_\varepsilon(k), A''_\varepsilon(k+1)) &= \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.
\end{align*}
\]

**Proof.** It is obvious that the sequence \(A_\varepsilon\) is strictly increasing on both coordinates. Moreover, for every \(k \in [n-1]\) we have \(\text{sl}(A_\varepsilon(k), A_\varepsilon(k+1)) = \varepsilon \text{sl}(A(k), A(k+1))\). Hence, \(A_\varepsilon\) is a south-east chain and we have \(\lim_{\varepsilon \to 0^+} \text{sl}(A_\varepsilon(k), A_\varepsilon(k+1)) = 0\). To prove the claim for the remaining two sequences, note that for sufficiently small \(\varepsilon > 0\), the inequality \(\text{sl}(A_\varepsilon(k), A_\varepsilon(k+1)) < \frac{1}{\sqrt{3}}\) is satisfied for all \(k \in [n-1]\).

Hence, by Lemma 2.3, the sequence \(A'_\varepsilon = \left(R(A_\varepsilon(1)), \ldots, R(A_\varepsilon(n))\right)\) is strictly increasing on both coordinates and the same is true for the sequence \(A''_\varepsilon = \left(\frac{A_\varepsilon(1) + A'_\varepsilon(1)}{2}, \ldots, \frac{A_\varepsilon(n) + A'_\varepsilon(n)}{2}\right)\). Moreover, if we denote \(\theta^{(k,\varepsilon)} := \arctan\left(\text{sl}(A_\varepsilon(k), A_\varepsilon(k+1))\right) < \frac{\pi}{6}\), then the sequence \(\theta^{(1,\varepsilon)}, \ldots, \theta^{(n-1,\varepsilon)}\) is strictly increasing and Lemma 2.3 shows that

\[\text{sl}(A'_\varepsilon(k), A'_\varepsilon(k+1)) = \tan\left(\frac{\pi}{3} + \theta^{(k,\varepsilon)}\right)\]
and
\[ \text{sl}(A'^i_e(k), A'^i_e(k+1)) = \tan(\frac{\pi}{6} + \theta(k, \varepsilon)). \]
In particular, the sequences \( A'_e, A''_e \) are south-east chains. Moreover, we have \( \lim_{\varepsilon \to 0^+} \theta(k, \varepsilon) = 0 \) for all \( k \), which gives the equalities
\[ \lim_{\varepsilon \to 0^+} \text{sl}(A'_e(k), A'_e(k+1)) = \tan(\frac{\pi}{3}) = \sqrt{3}, \]
\[ \lim_{\varepsilon \to 0^+} \text{sl}(A''_e(k), A''_e(k+1)) = \tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}. \] □

We now show how the transformations given in Lemma 2.4 can be used to prove Theorem 1.1. To do so, we take three south-east chains \( A := (A(1), \ldots, A(n)), B := (B(1), \ldots, B(n)), \) and \( C := (C(1), \ldots, C(m)) \) such that \( C \subset (A + B)/2 \). As discussed before, we let \( u := (1, 5/2), v := (0, 2), w := (1, 1) \) and we consider the chains \( A_e, A'_e, B_e, B'_e \) defined as in (1).

**Lemma 2.5.** If \( \varepsilon > 0 \) is sufficiently small, then the sequences \( A_e := (A_e, w + B'_e), B_e := (v + B_e, u + A'_e) \) are south-east chains. Moreover, the set \( (A_e + B_e)/2 \) contains a south-east chain of length at least \( 2m + n \).

In the statement above, \( w + B'_e \) denotes the sequence obtained by translating every element of \( B'_e \) by the vector \( w \) and \( (A_e, w + B'_e) \) denotes the concatenation of two sequences. The same applies to \( (v + B_e, u + A'_e) \).

**Proof of Lemma 2.5.** We start by proving that \( A_e \) is a south-east chain. By Lemma 2.4, the sequences \( A_e \) and \( B'_e \) are south-east chains for sufficiently small \( \varepsilon \). Hence, the sequence \( w + B'_e \) is also a south-east chain. Furthermore, we have \( \lim_{\varepsilon \to 0^+}(A_e(n)) = (0, 0) \) and \( \lim_{\varepsilon \to 0^+}(w + B'_e(1)) = w = (1, 1) \). In particular, for sufficiently small \( \varepsilon \), the sequence \( A_e \) is strictly increasing on both coordinates. Moreover, Lemma 2.4 shows the equalities \( \lim_{\varepsilon \to 0^+} \text{sl}(A_e(n-1), A_e(n)) = 0 \) and \( \lim_{\varepsilon \to 0^+} \text{sl}(w + B'_e(1), w + B'_e(2)) = \sqrt{3} \).

We also have
\[ \lim_{\varepsilon \to 0^+} \text{sl}(A_e(n), w + B'_e(1)) = \text{sl}(0, w) = 1. \]
Since \( 0 < 1 < \sqrt{3} \), the sequence \( A_e \) is a south-east chain for sufficiently small \( \varepsilon \). The proof for \( B_e \) is analogous—it is enough to observe that \( \lim_{\varepsilon \to 0^+}(v + B_e(n)) = (0, 2) \) and \( \lim_{\varepsilon \to 0^+}(u + A'_e(1)) = (1, 5/2) \) to show that \( B_e \) is strictly increasing on both coordinates. Then, \( B_e \) is a south-east chain by the equalities \( \lim_{\varepsilon \to 0^+} \text{sl}(v + B_e(n-1), v + B_e(n)) = 0 \), \( \lim_{\varepsilon \to 0^+} \text{sl}(u + A'_e(1), u + A'_e(2)) = \sqrt{3} \), and
\[ \lim_{\varepsilon \to 0^+} \text{sl}(v + B_e(n), u + A'_e(1)) = \text{sl}(v, u) = \frac{1}{2}. \]
It remains to show that \( (A_e + B_e)/2 \) contains a south-east chain of length at least \( 2m + n \). To do so, consider the chain \( C \subset (A + B)/2, |C| = m \), and let \( C_e, C'_e \) be defined as in (1). Furthermore, let \( t := (0, 1) = v/2 \) and \( z := (1, 7/4) = (u + w)/2 \). Since the transformations \( L_e \) and \( R \) are linear, we have \( t + C_e \subset (A_e + v + B_e)/2 \subset (A_e + B_e)/2 \) and \( z + C'_e \subset (u + A'_e + w + B'_e)/2 \subset (A_e + B_e)/2 \). Moreover, we define the chain \( A''_e \) as in (1) we let \( s := (1/2, 5/4) = u/2 \), and we note that \( s + A''_e \subset (A_e + u + \)
$A'_1/2 \subset (A_2 + B_2)/2$. Hence, the sequence $D_\varepsilon := (t + C_\varepsilon, s + A'_n, z + C'_n)$ is contained in $(A_2 + B_2)/2$ and its length is equal to $2m + n$. Therefore, it is enough to prove that $D_\varepsilon$ is a south-east chain. The proof is similar to the proofs before. By Lemma 2.4, the sequences $t + C_\varepsilon, s + A'_n$, and $z + C'_n$ are south-east chains for sufficiently small $\varepsilon$. Moreover, we have the equalities

\[ \lim_{\varepsilon \to 0^+} (t + C_\varepsilon(m)) = (0, 1), \quad \lim_{\varepsilon \to 0^+} (s + A'_n(1)) = (1/2, 5/4), \quad \text{and} \quad \lim_{\varepsilon \to 0^+} (z + C'_n(1)) = (1, 7/4), \]

which show that $D_\varepsilon$ is strictly increasing on both coordinates for sufficiently small $\varepsilon$. Furthermore, we use Lemma 2.4 to observe that $\lim_{\varepsilon \to 0^+} \text{sl}(t + C_\varepsilon(m - 1), t + C_\varepsilon(m)) = 0$, $\lim_{\varepsilon \to 0^+} \text{sl}(s + A'_n(1), s + A'_n(2)) = \lim_{\varepsilon \to 0^+} \text{sl}(s + A'_n(n - 1), s + A'_n(n)) = \frac{1}{\sqrt{3}}$, and $\lim_{\varepsilon \to 0^+} \text{sl}(z + C'_n(1), z + C'_n(2)) = \sqrt{3}$. To finish, we have

\[
\begin{align*}
\lim_{\varepsilon \to 0^+} \text{sl}(t + C_\varepsilon(m), s + A'_n(1)) &= \text{sl}(t, s) = \frac{1}{2}, \\
\lim_{\varepsilon \to 0^+} \text{sl}(s + A'_n(n), z + C'_n(1)) &= \text{sl}(s, z) = 1,
\end{align*}
\]

and $0 < \frac{1}{2} < \frac{1}{\sqrt{3}} < 1 < \sqrt{3}$. Hence, for sufficiently small $\varepsilon$, the sequence $D_\varepsilon$ is a south-east chain. 

As a corollary, we may prove our main theorem.

**Proof of Theorem 1.4.** As noted in the introduction, it is enough to prove the claim for $(P_k + Q_k)/2$ instead of $P_k + Q_k$. We show, by induction over $k$, that we can find two south-east chains $P_k, Q_k$, each of length $2^k$, such that $(P_k + Q_k)/2$ contains a south-east chain of length $(k + 2)2^{k-1}$. Let $P_1 := ((0, 0), (2, 1))$ and $Q_1 := ((0, 2), (2, 4))$. The sequences $P_1$ and $Q_1$ are south-east chains and the set $(P_1 + Q_1)/2$ contains a south-east chain of length three. Moreover, given $P_k, Q_k$ we may apply Lemma 2.5 to obtain two south-east chains $P_{k+1}, Q_{k+1}, k$ each of length $2^{k+1}$, such that $(P_{k+1} + Q_{k+1})/2$ contains a south-east chain of length at least $(k + 2)2^k + 2^k = (k + 3)2^k$. Therefore, the claim follows from Lemma 2.2.

\[ \square \]

3. **Final remarks**

Let us discuss some natural questions for further research. Firstly, we do not know how far from optimal is our construction. More precisely, consider the function $f : \mathbb{N}^* \to \mathbb{N}^*$ defined as

\[ f(n) := \max\{ci(P + Q) : P, Q \subset \mathbb{R}^2 \text{ are convexly independent and } |P| = |Q| = n\}. \]

By joining our analysis with the result of Tiwary [9], we obtain the asymptotic bound $f(n) = \Theta(n \log n)$. One can ask for an optimal constant $C$ such that $f(n) \leq Cn \log n$. Secondly, we wonder if our family of graphs $G_k = (U_k \cup V_k, E_k)$ can be still embedded in the plane if we impose a stronger condition on the midpoints of the edges. For instance, we may ask for an embedding such that $U_k$ and $V_k$ are south-east chains and the midpoints of $E_k$ are contained in some convex curve of small degree, such as a circle or a parabola. Our interest in this question is motivated by the following observation: if $G_k$ are realizable in the south-east quadrant of the plane in such a way that the midpoints of $E_k$ are on the unit circle, then we obtain a
Θ(\(n \log n\)) bound for the convex version of the unit distance problem, proposed by Erdős and Moser [4] (see [6, 1] for more information on lower and upper bounds for this variant of the unit distance problem). Indeed, suppose that \(h_k: (U_k \cup V_k) \rightarrow \mathbb{R}^2\) is an embedding of \(G_k\) such that \(h_k(U_k)\) and \(h_k(V_k)\) are south-east chains, \(h_k(U_k), h_k(V_k) \subset \{x \in \mathbb{R}^2: x_1 \geq 0, x_2 \leq 0\}\), and \(\|h_k(u)+h_k(v)\|_2 = 1\) for all \((u, v) \in E_k\). Then, the points \(-h_k(U_k)\) form a monotone concave chain in the north-west quadrant of the plane and \(h_k(V_k)\) form a monotone convex chain in the south-east quadrant of the plane, which implies that the collection \(\{-h_k(U_k), h_k(V_k)\}\) is convexly independent. Therefore, the polygon \(P_k := \text{conv}\{-h_k(U_k)/2, h_k(V_k)/2\}\) has \(2k+1\) vertices and \(\Theta(k2^k)\) diagonals of length one. As noted in the introduction, the existence of such an embedding cannot be excluded using the current list of forbidden matrices [1]. We were neither able to construct these embeddings nor find a new forbidden pattern that occurs in \(G_k\).

References