Convexly independent subsets of Minkowski sums of convex polygons

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CONVEXLY INDEPENDENT SUBSETS OF MINKOWSKI SUMS OF CONVEX POLYGONS

MATEUSZ SKOMRA[1] AND STÉPHAN THOMASSÉ[2,3]

Abstract. We show that there exist convex n-gons P and Q such that the largest convex polygon in the Minkowski sum $P+Q$ has size $\Theta(n \log n)$. This matches an upper bound of Tiwary.

1. Introduction

Let X be a finite set of points in the plane. A subset C of X is *convexly independent* if C forms a convex polygon. We denote by $\mathrm{ci}(X)$ the largest size of a convexly independent subset of X. The celebrated Happy Ending Theorem, from Erdős and Szekeres [5], asserts that $\mathrm{ci}(X)$ goes to infinity when $|X|$ goes to infinity and X does not have three points on the same line. More precisely, the minimum value one can achieve for $\mathrm{ci}(X)$ is logarithmic in $|X|$. To the opposite, one can try to maximize $\mathrm{ci}(X)$ when the set X satisfies some geometrical constraint. A well-studied case is when X is the *Minkowski sum* $P+Q \coloneqq \{p+q \colon p \in P, q \in Q\}$ of two sets of points P and Q.

Eisenbrand, Pach, Rothvoß, and Sopher [3] proved that $\mathrm{ci}(P+Q) = O(n^{4/3})$ when $|P| = |Q| = n$. This result was complemented by a construction of Bílka, Buchin, Fulek, Kiyomi, Okamoto, Tanigawa, and Tóth [2] showing the existence of such P and Q satisfying $\mathrm{ci}(P+Q) = \Theta(n^{4/3})$. Surprisingly, the set Q they use in the extremal constructions can be chosen convex. A natural question is then to ask for the maximum possible value of $\mathrm{ci}(P+Q)$ when both P and Q are convex polygons. In 2014, Tiwary [9] proposed an upper bound by showing that $\mathrm{ci}(P+Q) = O\big((n+m)\log(n+m)\big)$ when P and Q are respectively a convex n-gon and a convex m-gon. He concluded his paper by mentioning that his upper bound seemed very generous, and left as an open problem the existence of a matching lower bound. Our main result in this paper is that Tiwary's proof indeed provides a sharp bound by exhibiting a matching construction.

Theorem 1.1. *There exist two families $(P_k)_{k \geqslant 1}, (Q_k)_{k \geqslant 1} \subset \mathbb{R}^2$ of convexly independent sets such that $|P_k| = |Q_k| = 2^k$ and $\mathrm{ci}(P_k + Q_k) \geqslant (k+2)2^{k-1}$ for all $k \geqslant 1$.*

An equivalent point of view of Minkowski sums is to consider $(P+Q)/2$ instead of $P+Q$, and therefore $\mathrm{ci}(P+Q)$ represents the maximum size of a set M of midpoints of P, Q-segments in convex position. If we actually draw

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all segments corresponding to these midpoints, we get a bipartite graph with vertex set $P \uplus Q$, whose edges are all pairs $(p, q) \in P \times Q$ such that $(p+q)/2 \in M$. In our construction, the number of edges of this graph is $cn \log n$, all vertices of P and Q are in convex position, and all midpoints of edges are also in convex position. This kind of drawing was introduced by Halman, Onn, and Rothblum [8] where they define a *strong convex embedding* of a graph $G = (V, E)$ as a function $f \colon V \to \mathbb{R}^2$ such that $f(V)$ is convex and $\{(f(x) + f(y))/2 \colon (x, y) \in E\}$ is also convex. They showed that if G admits a strong convex embedding, then $|E| \leqslant 5n - 8$ when $n \geqslant 3$ is the number of vertices. Recently, García-Marco and Knauer [7] reduced this bound to $2n - 3$. Equivalently their result shows that $\mathrm{ci}(P + P) \leqslant 2n - 3$ when P is a convex n-gon. Perhaps surprisingly, our construction shows that when slightly relaxing strong convex embedding to only ask convex positions for the two partite sets of a bipartite graph, the bound goes from linear to $n \log n$.

Another very attractive reason motivating the study of $\mathrm{ci}(P + Q)$ for convex n-gons is the famous unit distance problem of Erdős and Moser [4] asking for the maximum number of pairs of points in a convex n-gon P with distance exactly one. The trick is to observe that if two points p, p' of P have distance 1, then $p - p'$ lies on the unit circle. A careful counting shows, in particular, that $\mathrm{ci}\big(P + (-P)\big)$ is an upper bound on the number of unit distance pairs. Unfortunately, our construction shows that $\mathrm{ci}(P+Q)$ cannot be directly used to improve the already known $O(n \log n)$ upper bound on the unit distance problem (see Füredi [6] and Aggarwal [1] for a sharper estimate). We do not know however if the graphs G_k arising from our construction can be realized as convex unit distance graphs. If not, it would be interesting to find a forbidden pattern in G_k. This would extend the list of forbidden matrices provided in [1], as none of them appears in G_k.

2. Proof of the main theorem

The proof of Tiwary [9] is based on the fact that any collection of convexly independent points in $\mathbb{R}^2$ can be split into at most four monotone chains as in Figure 1. We use the name "south-east chain" for the chain that is situated in the bottom-right part of this figure. Our construction is based on this type of chains. More formally, we make the following definition and point out the subsequent lemma.

Definition 2.1. A *south-east chain* is a sequence $(a^{(1)}, \dots, a^{(n)}) \subset \mathbb{R}^2$ of $n \geqslant 2$ points in the plane that satisfies the following two conditions. First, the sequence is strictly increasing on both coordinates, i.e., we have $a_1^{(1)} < a_1^{(2)} < \dots < a_1^{(n)}$ and $a_2^{(1)} < a_2^{(2)} < \dots < a_2^{(n)}$ (where $a_i^{(k)}$ denotes the ith coordinate of the point $a^{(k)}$). Second, the corresponding sequence of consecutive slopes is strictly increasing, i.e., we have

$$\frac{a_2^{(2)} - a_2^{(1)}}{a_1^{(2)} - a_1^{(1)}} < \dots < \frac{a_2^{(k+1)} - a_2^{(k)}}{a_1^{(k+1)} - a_1^{(k)}} < \dots < \frac{a_2^{(n)} - a_2^{(n-1)}}{a_1^{(n)} - a_1^{(n-1)}} \, .$$

We say that n is the *length* of a south-east chain $(a^{(1)}, \dots, a^{(n)})$.

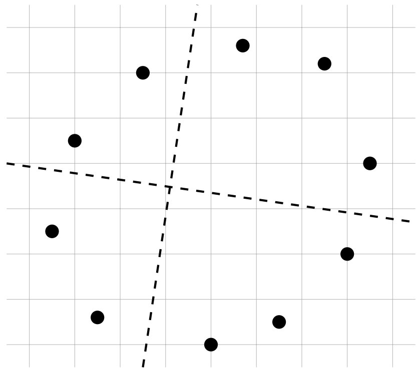

FIGURE 1. Splitting convexly independent points into monotone chains.

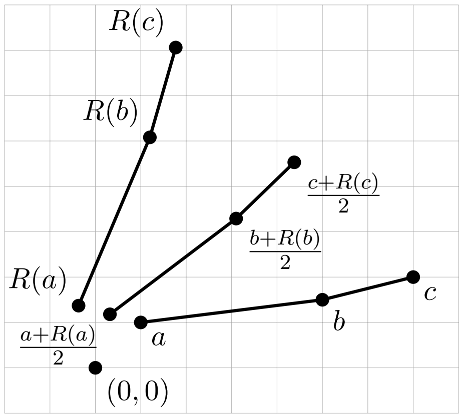

FIGURE 2. Rotation of a flat south-east chain.

Lemma 2.2. *If the points* $(a^{(1)}, \dots, a^{(n)})$ *form a south-east chain, then they are convexly independent, i.e., the convex hull of* $\{a^{(1)}, \dots, a^{(n)}\}$ *has* n *vertices.*

Before giving a formal proof of Theorem 1.1, let us explain it intuitively. Our proof is based on some elementary properties of rotations. Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the counterclockwise rotation by 60 degrees centered at zero. Let $a, b, c \in \mathbb{R}^2$ be three points on a plane such that (a, b, c) is a south-east chain. Moreover, suppose that this chain is sufficiently flat (more precisely, that the line defined by extending the segment $[b, c]$ forms an angle smaller than 30 degrees with the horizontal axis). Then, the images $(R(a), R(b), R(c))$ also form a south-east chain. Moreover, if all the slopes between consecutive points of (a, b, c) are close to 0, then the corresponding slopes of $(R(a), R(b), R(c))$ are close to $\sqrt{3}$ (in other words, the corresponding segments form angles close to 60 degrees with the horizontal axis). Even more, under these conditions, the triple $\frac{1}{2}(a + R(a), b + R(b), c + R(c))$ also forms a south-east chain, and its slopes are close to ${}^{1}/\sqrt{3}$ (which corresponds to the angle of 30 degrees). We refer to Figure 2 for an illustration. The same applies to south-east chains formed by more than three points. In

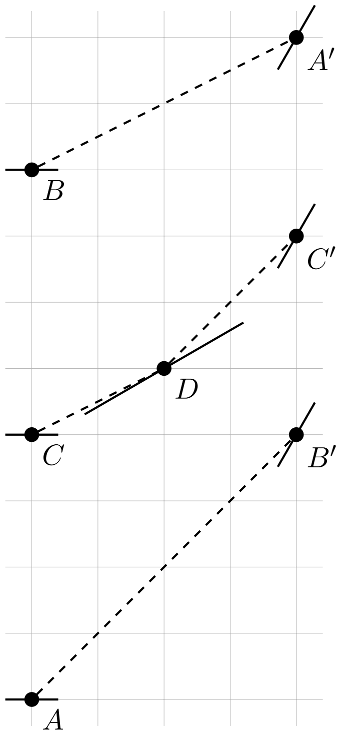

FIGURE 3. The construction of south-east chains.

this way, given a sufficiently flat south-east chain, we can construct two new chains, one with slopes close to $\sqrt{3}$ and one with slopes close to $1/\sqrt{3}$.

Consider now the linear map $L_\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}^2$, $L_\varepsilon(x, y) := (\varepsilon x, \varepsilon^2 y)$, where $\varepsilon > 0$. It is immediate to see that this map preserves south-east chains (i.e., an image of a south-east chain under L_ε is again a south-east chain). Moreover, this map flattens the chains. In other words, if A is a south-east chain and $\varepsilon > 0$ is small enough, then the image of A under L_ε is a south-east chain that is sufficiently flat to apply the previous observations. Moreover, for small ε, the image of A under L_ε is contained in a small neighborhood of 0.

Suppose now that we are given three south-east chains A, B, C such that C is included in the set of points $(A + B)/2$. By applying L_ε to all three chains, we can suppose that they are arbitrarily flat, and contained in a small neighborhood of 0. Then, we can apply the rotation R to A, B, C. In this way, we obtain three chains A', B', C' that are again contained in a neighborhood of 0, but whose slopes are close to $\sqrt{3}$. We now translate the chains as follows. We do not apply any translation to A, we translate B by the vector $(0, 2)$, and C by the vector $(0, 1)$. Then, we translate A' by the vector $(1, 5/2)$, B' by the vector $(1, 1)$, and C' by the vector $(1, 7/4)$. This gives the situation depicted in Figure 3. In this picture, the chain A is contained in the small neighborhood of the point marked by A, and the slopes of this chain are close to the slope of the solid line passing through A. The same is true for the chains B, C, A', B', C'. In particular, for sufficiently

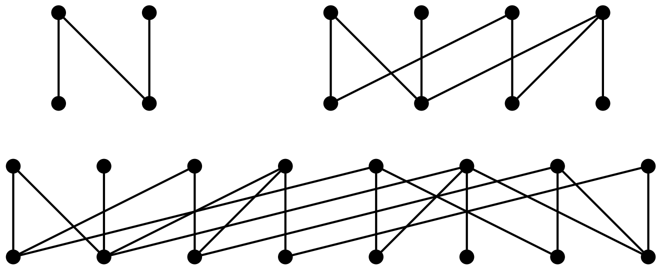

FIGURE 4. Graphs that can be drawn using south-east chains.

small ε, the concatenation $\overline{A} \coloneqq (A, B')$ of chains A and B' forms a south-east chain (because the dashed line from A to B' has slope greater than the slope of the solid line passing through A but smaller than the slope of the solid line passing through B'). By the same reasoning, the concatenation $\overline{B} \coloneqq (B, A')$ forms a south-east chain. Denote $A = (a^{(1)}, \dots, a^{(n)})$ and $A' = (a'^{(1)}, \dots, a'^{(n)})$. By the observation about rotation made above, the sequence $D \coloneqq (\frac{a^{(1)}+a'^{(1)}}{2}, \dots, \frac{a^{(n)}+a'^{(n)}}{2})$ is a south-east chain, and all the slopes of this chain are close to $1/\sqrt{3}$. Moreover, this chain is contained in a small neighborhood of the point $(1/2, 5/4)$. We marked this chain in Figure 3 using the same conventions as for the remaining chains. By applying the same reasoning as above, the concatenation $\overline{C} \coloneqq (C, D, C')$ is a south-east chain. To summarize, our construction shows the following statement. Given three south-east chains A, B, C such that C is included in the set of points $(A + B)/2$, we can construct three south-east chains $\overline{A}, \overline{B}, \overline{C}$ such that $\overline{C}$ is included in $(\overline{A} + \overline{B})/2$ and $|\overline{A}| = |\overline{B}| = |A| + |B|$, $|\overline{C}| = 2|C| + |A|$. Thus, if we suppose that $|A| = |B| = n$, then we have $|\overline{A}| = |\overline{B}| = 2n$ and $|\overline{C}| = 2|C| + n$. By iterating this reasoning, we obtain the claimed bound $\Theta(n \log n)$.

Before presenting a formal proof, let us discuss the types of graph drawings that we obtain in this way. Here, we are interested in a drawing $f\colon (U \uplus V) \to \mathbb{R}^2$ of a bipartite graph $G = (U \uplus V, E)$ such that $f(U)$ is a south-east chain, $f(V)$ is a south-east chain, and the midpoints $\{(f(u) + f(v))/2 \colon (u, v) \in E\}$ also form a south-east chain. The construction described above implies that if G is drawable in this way and $G' \coloneqq (U' \uplus V', E')$ is a copy of G, then the graph $\overline{G} \coloneqq (\overline{U} \uplus \overline{V}, \overline{E})$ defined as $\overline{U} \coloneqq U \uplus V'$, $\overline{V} \coloneqq V \uplus U'$, and

$$\overline{E} \coloneqq E \uplus E' \uplus \{(u, u') \colon u \in U\}$$

is also drawable in this fashion. By starting from a graph

$$G_1 = (\{u_1, u_2\} \uplus \{v_1, v_2\}, \{(u_1, v_1), (u_2, v_1), (u_2, v_2)\})$$

and iterating the procedure, we obtain a family $(G_k)_{k \geqslant 1}$ of drawable graphs, each having 2^{k+1} vertices and $(k+2)2^{k-1}$ edges. Figure 4 depicts this family of graphs for $k \leqslant 3$ and Figure 5 depicts a drawing of G_3 using south-east chains.

In the remaining part of this section, we give a formal proof of the argument described above. Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the counterclockwise rotation by 60 degrees centered at zero, i.e., the linear transformation given

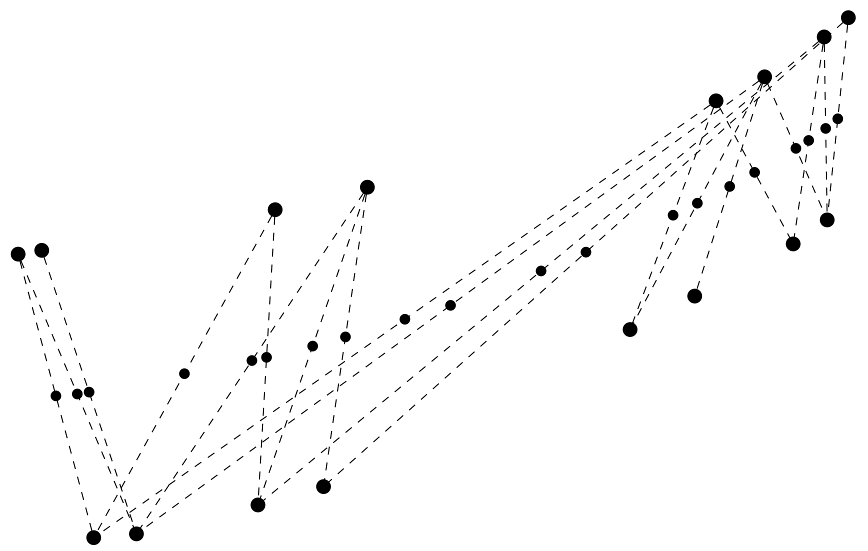

FIGURE 5. Drawing of G_3 using south-east chains.

by the matrix

$$R := \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.$$

Furthermore, if $a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2$ are two points such that $a_1 < b_1$ and $a_2 < b_2$, then we denote by

$$\mathrm{sl}(a, b) := \frac{b_2 - a_2}{b_1 - a_1}$$

the corresponding slope of the segment $[a, b]$. The following lemma, which can be proven using elementary trigonometric identities, gathers the properties of rotation that were mentioned above.

Lemma 2.3. *Suppose that two points* $a := (a_1, a_2), b := (b_1, b_2) \in \mathbb{R}^2$ *are such that* $a_1 < b_1$, $a_2 < b_2$, *and* $\mathrm{sl}(a, b) < \frac{1}{\sqrt{3}} = \tan(\frac{\pi}{6})$. *Let* $\theta := \arctan\big(\mathrm{sl}(a, b)\big) < \frac{\pi}{6}$ *and denote* $\tilde{a} = R(a)$, $\tilde{b} = R(b)$. *Then, we have* $\tilde{a}_1 < \tilde{b}_1$, $\tilde{a}_2 < \tilde{b}_2$,

$$\mathrm{sl}(\tilde{a}, \tilde{b}) = \tan(\frac{\pi}{3} + \theta), \quad \text{and} \quad \mathrm{sl}\Big(\frac{a + \tilde{a}}{2}, \frac{b + \tilde{b}}{2}\Big) = \tan(\frac{\pi}{6} + \theta).$$

Proof. We have $\tilde{a} = \frac{1}{2}(a_1 - \sqrt{3}a_2, \sqrt{3}a_1 + a_2)$ and $\tilde{b} = \frac{1}{2}(b_1 - \sqrt{3}b_2, \sqrt{3}b_1 + b_2)$. The inequality $\tilde{a}_2 < \tilde{b}_2$ is trivial. Moreover, $\tilde{b}_1 > \tilde{a}_1 \iff b_1 - a_1 > \sqrt{3}(b_2 - a_2)$, which is true by our assumptions. Furthermore, we have

$$\begin{aligned}\tan(\frac{\pi}{3} + \theta) &= \frac{\tan(\frac{\pi}{3}) + \tan(\theta)}{1 - \tan(\frac{\pi}{3})\tan(\theta)} = \frac{\sqrt{3}(b_1 - a_1) + b_2 - a_2}{b_1 - a_1 - \sqrt{3}(b_2 - a_2)} \\ &= \frac{\tilde{b}_2 - \tilde{a}_2}{\tilde{b}_1 - \tilde{a}_1} = \mathrm{sl}(\tilde{a}, \tilde{b}).\end{aligned}$$

Similarly,

$$\tan(\frac{\pi}{6}+\theta) = \frac{\tan(\frac{\pi}{6})+\tan(\theta)}{1-\tan(\frac{\pi}{6})\tan(\theta)} = \frac{b_1-a_1+\sqrt{3}(b_2-a_2)}{\sqrt{3}(b_1-a_1)-(b_2-a_2)}$$
$$= \frac{b_2+\tilde{b}_2-a_2-\tilde{a}_2}{b_1+\tilde{b}_1-a_1-\tilde{a}_1} = \mathrm{sl}\Big(\frac{a+\tilde{a}}{2},\frac{b+\tilde{b}}{2}\Big). \qquad \square$$

For any $\varepsilon > 0$ we denote by $L_\varepsilon\colon \mathbb{R}^2 \to \mathbb{R}^2$ the linear transformation $L_\varepsilon(x,y) := (\varepsilon x, \varepsilon^2 y)$. As a corollary of Lemma 2.3 we may now prove the properties of the three transformations of south-east chains discussed above. To improve readability, we use the following notation: if A is a sequence, then we denote by $A(k)$ its kth element, so that $A = \big(A(1),\dots,A(n)\big)$. If A is a south-east chain of length n and $\varepsilon > 0$, then we consider the following three sequences:

$$\begin{aligned} A_\varepsilon &:= \Big(L_\varepsilon(A(1)),\dots,L_\varepsilon(A(n))\Big)\,,\\ A'_\varepsilon &:= \Big(R(A_\varepsilon(1)),\dots,R(A_\varepsilon(n))\Big)\,,\\ A''_\varepsilon &:= \Big(\frac{A_\varepsilon(1)+A'_\varepsilon(1)}{2},\dots,\frac{A_\varepsilon(n)+A'_\varepsilon(n)}{2}\Big)\,. \end{aligned} \tag{1}$$

Using this notation, A_ε is a chain obtained by flattening A, A'_ε is the rotated version of this flattened chain, and A''_ε is the chain formed by taking the midpoints of the two previous chains. The next result follows from Lemma 2.3.

Lemma 2.4. *Suppose that $A := \big(A(1),\dots,A(n)\big)$ is a south-east chain. Then, for sufficiently small $\varepsilon > 0$, the sequences $A_\varepsilon, A'_\varepsilon, A''_\varepsilon$ are south-east chains. Moreover, for every $k \in [n-1]$ we have the equalities*

$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A_\varepsilon(k),A_\varepsilon(k+1)\big) = 0\,,$$
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A'_\varepsilon(k),A'_\varepsilon(k+1)\big) = \tan(\frac{\pi}{3}) = \sqrt{3}\,,$$
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A''_\varepsilon(k),A''_\varepsilon(k+1)\big) = \tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}\,.$$

Proof. It is obvious that the sequence A_ε is strictly increasing on both coordinates. Moreover, for every $k \in [n-1]$ we have $\mathrm{sl}\big(A_\varepsilon(k),A_\varepsilon(k+1)\big) = \varepsilon\,\mathrm{sl}\big(A(k),A(k+1)\big)$. Hence, A_ε is a south-east chain and we have $\lim_{\varepsilon\to 0^+}\mathrm{sl}\big(A_\varepsilon(k),A_\varepsilon(k+1)\big) = 0$. To prove the claim for the remaining two sequences, note that for sufficiently small $\varepsilon > 0$, the inequality $\mathrm{sl}\big(A_\varepsilon(k),A_\varepsilon(k+1)\big) = \varepsilon\,\mathrm{sl}\big(A(k),A(k+1)\big) < 1/\sqrt{3}$ is satisfied for all $k \in [n-1]$. Hence, by Lemma 2.3, the sequence $A'_\varepsilon = \Big(R(A_\varepsilon(1)),\dots,R(A_\varepsilon(n))\Big)$ is strictly increasing on both coordinates and the same is true for the sequence $A''_\varepsilon = \Big(\frac{A_\varepsilon(1)+A'_\varepsilon(1)}{2},\dots,\frac{A_\varepsilon(n)+A'_\varepsilon(n)}{2}\Big)$. Moreover, if we denote $\theta^{(k,\varepsilon)} := \arctan\Big(\mathrm{sl}\big(A_\varepsilon(k),A_\varepsilon(k+1)\big)\Big) < \frac{\pi}{6}$, then the sequence $\theta^{(1,\varepsilon)},\dots,\theta^{(n-1,\varepsilon)}$ is strictly increasing and Lemma 2.3 shows that

$$\mathrm{sl}\big(A'_\varepsilon(k),A'_\varepsilon(k+1)\big) = \tan(\frac{\pi}{3}+\theta^{(k,\varepsilon)})$$

and
$$\mathrm{sl}\big(A''_\varepsilon(k), A''_\varepsilon(k+1)\big) = \tan\big(\frac{\pi}{6} + \theta^{(k,\varepsilon)}\big)\,.$$
In particular, the sequences $A'_\varepsilon, A''_\varepsilon$ are south-east chains. Moreover, we have $\lim_{\varepsilon\to 0^+} \theta^{(k,\varepsilon)} = 0$ for all k, which gives the equalities
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A'_\varepsilon(k), A'_\varepsilon(k+1)\big) = \tan\big(\frac{\pi}{3}\big) = \sqrt{3}\,,$$
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A''_\varepsilon(k), A''_\varepsilon(k+1)\big) = \tan\big(\frac{\pi}{6}\big) = \frac{1}{\sqrt{3}}\,. \qquad \square$$

We now show how the transformations given in Lemma 2.4 can be used to prove Theorem 1.1. To do so, we take three south-east chains $A \coloneqq \big(A(1),\dots,A(n)\big)$, $B \coloneqq \big(B(1),\dots,B(n)\big)$, and $C \coloneqq \big(C(1),\dots,C(m)\big)$ such that $C \subset (A+B)/2$. As discussed before, we let $u \coloneqq (1, 5/2)$, $v \coloneqq (0,2)$, $w \coloneqq (1,1)$ and we consider the chains $A_\varepsilon, A'_\varepsilon, B_\varepsilon, B'_\varepsilon$ defined as in (1).

Lemma 2.5. *If $\varepsilon > 0$ is sufficiently small, then the sequences $\overline{A}_\varepsilon \coloneqq (A_\varepsilon, w + B'_\varepsilon)$, $\overline{B}_\varepsilon \coloneqq (v + B_\varepsilon, u + A'_\varepsilon)$ are south-east chains. Moreover, the set $(\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ contains a south-east chain of length at least $2m+n$.*

In the statement above, $w + B'_\varepsilon$ denotes the sequence obtained by translating every element of B'_ε by the vector w and $(A_\varepsilon, w + B'_\varepsilon)$ denotes the concatenation of two sequences. The same applies to $(v + B_\varepsilon, u + A'_\varepsilon)$.

Proof of Lemma 2.5. We start by proving that $\overline{A}_\varepsilon$ is a south-east chain. By Lemma 2.4, the sequences A_ε and B'_ε are south-east chains for sufficiently small ε. Hence, the sequence $w + B'_\varepsilon$ is also a south-east chain. Furthermore, we have $\lim_{\varepsilon\to 0^+}\big(A_\varepsilon(n)\big) = (0,0)$ and $\lim_{\varepsilon\to 0^+}\big(w + B'_\varepsilon(1)\big) = w = (1,1)$. In particular, for sufficiently small ε, the sequence $\overline{A}_\varepsilon$ is strictly increasing on both coordinates. Moreover, Lemma 2.4 shows the equalities $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A_\varepsilon(n-1), A_\varepsilon(n)\big) = 0$ and $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(w + B'_\varepsilon(1), w + B'_\varepsilon(2)\big) = \sqrt{3}$. We also have
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(A_\varepsilon(n), w + B'_\varepsilon(1)\big) = \mathrm{sl}(0, w) = 1\,.$$
Since $0 < 1 < \sqrt{3}$, the sequence $\overline{A}_\varepsilon$ is a south-east chain for sufficiently small ε. The proof for $\overline{B}_\varepsilon$ is analogous—it is enough to observe that $\lim_{\varepsilon\to 0^+}\big(v + B_\varepsilon(n)\big) = (0,2)$ and $\lim_{\varepsilon\to 0^+}\big(u + A'_\varepsilon(1)\big) = (1, 5/2)$ to show that $\overline{B}_\varepsilon$ is strictly increasing on both coordinates. Then, $\overline{B}_\varepsilon$ is a south-east chain by the equalities $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(v + B_\varepsilon(n-1), v + B_\varepsilon(n)\big) = 0$, $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(u + A'_\varepsilon(1), u + A'_\varepsilon(2)\big) = \sqrt{3}$, and
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(v + B_\varepsilon(n), u + A'_\varepsilon(1)\big) = \mathrm{sl}(v, u) = \frac{1}{2}\,.$$
It remains to show that $(\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ contains a south-east chain of length at least $2m+n$. To do so, consider the chain $C \subset (A+B)/2$, $|C| = m$, and let $C_\varepsilon, C'_\varepsilon$ be defined as in (1). Furthermore, let $t \coloneqq (0,1) = v/2$ and $z \coloneqq (1, 7/4) = (u+w)/2$. Since the transformations L_ε and R are linear, we have $t + C_\varepsilon \subset (A_\varepsilon + v + B_\varepsilon)/2 \subset (\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ and $z + C'_\varepsilon \subset (u + A'_\varepsilon + w + B'_\varepsilon)/2 \subset (\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$. Moreover, we define the chain A''_ε as in (1), we let $s \coloneqq (1/2, 5/4) = u/2$, and we note that $s + A''_\varepsilon \subset (A_\varepsilon + u +$

$A'_\varepsilon)/2 \subset (\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$. Hence, the sequence $D_\varepsilon := (t + C_\varepsilon, s + A''_\varepsilon, z + C'_\varepsilon)$ is contained in $(\overline{A}_\varepsilon + \overline{B}_\varepsilon)/2$ and its length is equal to $2m + n$. Therefore, it is enough to prove that D_ε is a south-east chain. The proof is similar to the proofs before. By Lemma 2.4, the sequences $t + C_\varepsilon$, $s + A''_\varepsilon$, and $z + C'_\varepsilon$ are south-east chains for sufficiently small ε. Moreover, we have the equalities $\lim_{\varepsilon\to 0^+}\big(t + C_\varepsilon(m)\big) = (0, 1)$, $\lim_{\varepsilon\to 0^+}\big(s + A''_\varepsilon(1)\big) = \lim_{\varepsilon\to 0^+}\big(s + A''_\varepsilon(n)\big) = (1/2, 5/4)$, and $\lim_{\varepsilon\to 0^+}\big(z + C'_\varepsilon(1)\big) = (1, 7/4)$, which show that D_ε is strictly increasing on both coordinates for sufficiently small ε. Furthermore, we use Lemma 2.4 to observe that $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(t + C_\varepsilon(m-1), t + C_\varepsilon(m)\big) = 0$, $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(s+A''_\varepsilon(1), s+A''_\varepsilon(2)\big) = \lim_{\varepsilon\to 0^+} \mathrm{sl}\big(s+A''_\varepsilon(n-1), s+A''_\varepsilon(n)\big) = \frac{1}{\sqrt{3}}$, and $\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(z + C'_\varepsilon(1), z + C'_\varepsilon(2)\big) = \sqrt{3}$. To finish, we have

$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(t + C_\varepsilon(m), s + A''_\varepsilon(1)\big) = \mathrm{sl}(t, s) = \frac{1}{2}\,,$$
$$\lim_{\varepsilon\to 0^+} \mathrm{sl}\big(s + A''_\varepsilon(n), z + C'_\varepsilon(1)\big) = \mathrm{sl}(s, z) = 1\,,$$

and $0 < \frac{1}{2} < \frac{1}{\sqrt{3}} < 1 < \sqrt{3}$. Hence, for sufficiently small ε, the sequence D_ε is a south-east chain. □

As a corollary, we may prove our main theorem.

Proof of Theorem 1.1. As noted in the introduction, it is enough to prove the claim for $(P_k + Q_k)/2$ instead of $P_k + Q_k$. We show, by induction over k, that we can find two south-east chains P_k, Q_k, each of length 2^k, such that $(P_k + Q_k)/2$ contains a south-east chain of length $(k + 2)2^{k-1}$. Let $P_1 := \big((0, 0), (2, 1)\big)$ and $Q_1 := \big((0, 2), (2, 4)\big)$. The sequences P_1 and Q_1 are south-east chains and the set $(P_1 + Q_1)/2$ contains a south-east chain of length three. Moreover, given P_k, Q_k we may apply Lemma 2.5 to obtain two south-east chains P_{k+1}, Q_{k+1}, each of length 2^{k+1}, such that $(P_{k+1}+Q_{k+1})/2$ contains a south-east chain of length at least $(k + 2)2^k + 2^k = (k + 3)2^k$. Therefore, the claim follows from Lemma 2.2. □

3. Final remarks

Let us discuss some natural questions for further research. Firstly, we do not know how far from optimal is our construction. More precisely, consider the function $f\colon \mathbb{N}^* \to \mathbb{N}^*$ defined as

$$f(n) := \max\{\mathrm{ci}(P + Q)\colon P, Q \subset \mathbb{R}^2 \text{ are convexly independent and } |P| = |Q| = n\}\,.$$

By joining our analysis with the result of Tiwary [9], we obtain the asymptotic bound $f(n) = \Theta(n \log n)$. One can ask for an optimal constant C such that $f(n) \leqslant Cn \log n$. Secondly, we wonder if our family of graphs $G_k = (U_k \uplus V_k, E_k)$ can be still embedded in the plane if we impose a stronger condition on the midpoints of the edges. For instance, we may ask for an embedding such that U_k and V_k are south-east chains and the midpoints of E_k are contained in some convex curve of small degree, such as a circle or a parabola. Our interest in this question is motivated by the following observation: if G_k are realizable in the south-east quadrant of the plane in such a way that the midpoints of E_k are on the unit circle, then we obtain a

$\Theta(n \log n)$ bound for the convex version of the unit distance problem, proposed by Erdős and Moser [4] (see [6, 1] for more information on lower and upper bounds for this variant of the unit distance problem). Indeed, suppose that $h_k\colon (U_k \uplus V_k) \to \mathbb{R}^2$ is an embedding of G_k such that $h_k(U_k)$ and $h_k(V_k)$ are south-east chains, $h_k(U_k), h_k(V_k) \subset \{x \in \mathbb{R}^2 \colon x_1 \geqslant 0, x_2 \leqslant 0\}$, and $\|\frac{h_k(u)+h_k(v)}{2}\|_2 = 1$ for all $(u,v) \in E_k$. Then, the points $-h_k(U_k)$ form a monotone concave chain in the north-west quadrant of the plane and $h_k(V_k)$ form a monotone convex chain in the south-east quadrant of the plane, which implies that the collection $\{-h_k(U_k), h_k(V_k)\}$ is convexly independent. Therefore, the polygon $P_k \coloneqq \mathrm{conv}\{-h_k(U_k)/2, h_k(V_k)/2\}$ has 2^{k+1} vertices and $\Theta(k2^k)$ diagonals of length one. As noted in the introduction, the existence of such an embedding cannot be excluded using the current list of forbidden matrices [1]. We were neither able to construct these embeddings nor find a new forbidden pattern that occurs in G_k.

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