L 2+ Induced Norm Analysis of Continuous-Time LTI Systems Using Positive Filters and Copositive Programming

Yoshio Ebihara, Hayato Waki, Noboru Sebe, Victor Magron, Dimitri Peaucelle, Sophie Tarbouriech

L_{2+} Induced Norm Analysis of Continuous-Time LTI Systems Using Positive Filters and Copositive Programming

Yoshio Ebihara, Hayato Waki, Noboru Sebe,
Victor Magron, Dimitri Peaucelle, and Sophie Tarbouriech

***Abstract*— This paper is concerned with the analysis of the L_2 induced norm of continuous-time LTI systems where the input signals are restricted to be nonnegative. This induced norm is referred to as the L_{2+} induced norm in this paper. It has been shown very recently that the L_{2+} induced norm is particularly useful for the stability analysis of nonlinear feedback systems constructed from linear systems and static nonlinearities where the nonlinear elements only provide nonnegative signals. For the upper bound computation of the L_{2+} induced norm, an approach with copositive programming has also been proposed. It is nonetheless true that this approach becomes effective only for multi-input systems, and for single-input systems this approach does not bring any improvement over the trivial upper bound, the standard L_2 norm. To overcome this difficulty, we newly introduce positive filters to increase the number of positive signals. This enables us to enlarge the size of the copositive multipliers so that we can obtain better (smaller) upper bounds with copositive programming.**
Keywords: nonnegative signal, L_{2+} induced norm, positive filter, copositive programming

I. INTRODUCTION

Recently, control theoretic approaches for the analysis of control systems driven by neural networks (NNs) have attracted great attention [1], [2], [3]. The basic treatment there is to recast the control system of interest into a feedback system constructed from a linear system and nonlinear activation functions. Then, by capturing the properties of nonlinear activation functions with the integral quadratic constraint (IQC) framework [4], we can obtain numerically tractable semidefinite programming problems (SDPs) for the analysis.

On the other hand, some activation functions in NNs exhibit particular nonnegative properties that are hardly captured by the standard IQC framework on the positive semidefinite cone. As a typical example, the rectified linear units (ReLUs) return only nonnegative output signals irrespective of input signals. This is the motivation of [5] to consider the L_2 induced norm of continuous-time LTI systems where the input signals are restricted to be nonnegative. This induced norm is referred to as the L_{2+} induced norm in this paper. As the main result of [5], an L_{2+}-induced-norm-based small gain theorem has been derived for the stability analysis of recurrent neural networks with activation functions being ReLUs. This has been proved to be less conservative than the standard L_2-induced-norm-based small gain theorem [6]. Moreover, for the upper bound computation of the L_{2+} induced norm, an approach with copositive programming (COP) has also been proposed in [5]. By applying inner approximation to the copositive cone, we can eventually obtain numerically tractable SDPs for the upper bound computation.

Even though the preceding work [5] provides basic ideas for the treatment of the L_{2+} induced norm, the results there are certainly deficient in the following aspects:

(i) For single-input systems, the upper bound characterized by the COP in [5] reduces to the trivial upper bound, the L_2 induced norm. Namely, it is by no means possible to obtain better upper bounds than the trivial one.

(ii) For multi-input systems, we can obtain a better upper bound than the L_2 induced norm by solving an SDP in [5]. However, once we obtain this upper bound, there is no way to obtain further better upper bounds.

These deficiencies are related to the size of the copositive multipliers that is introduced in [5] to capture the nonnegativity of the input signals. If we can somehow increase the number of nonnegative signals, then we can enlarge the size (freedom) of the corresponding copositive multiplier so that we can obtain better (smaller) upper bounds.

To achieve this end, in this paper, we newly introduce positive filters to increase the number of nonnegative signals and then enlarge the size of copositive multipliers. More precisely, we introduce a positive filter of specific form. By increasing the degree of the positive filter, we can construct a sequence of COPs and then a sequence of SDPs by applying inner approximation to the copositive cone. We prove that, by solving the sequence of SDPs, we can construct a monotonically nonincreasing sequence of the upper bounds of the L_{2+} induced norm. The effectiveness of the proposed positive-filter-based method is illustrated by numerical examples. We finally note that the analysis of the L_{2+} induced norm is also motivated by recent advancement on the study of positive systems [7], [8], [9], [10], [11], where the treatment of nonnegative signals is essentially important.

Notation: The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$, and the set of $n \times m$ entrywise nonnegative matrices is denoted by $\mathbb{R}_+^{n\times m}$. For a matrix A, we also write $A \geq 0$ ($A > 0$) to denote that A is entrywise nonnegative (positive). We denote the set of $n \times n$ real symmetric, positive semidefinite, and positive definite matrices by $\mathbb{S}^n$,

Y. Ebihara is with the Graduate School of Information Science and Electrical Engineering, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan, he was also with LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France, in 2011. H. Waki is with the Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan. N. Sebe is with the Department of Intelligent and Control Systems, Kyushu Institute of Technology, Fukuoka 820-8502, Japan. V. Magron, D. Peaucelle, and S. Tarbouriech are with LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France.

$\mathbb{S}_+^n$, and $\mathbb{S}_{++}^n$, respectively. The set of $n \times n$ Hurwitz and Metzler matrices are denoted by $\mathbb{H}^n$ and $\mathbb{M}^n$, respectively. For $A \in \mathbb{S}^n$, we also write $A \succ 0$ ($A \prec 0$) to denote that A is positive (negative) definite.

II. Preliminaries

A. Norms for Signals and Systems

For a continuous-time signal w defined over the time interval $[0, \infty)$, we define

$$\|w\|_2 := \sqrt{\int_0^\infty |w(t)|_2^2 dt}$$

where for $v \in \mathbb{R}^{n_v}$ we define $|v|_2 := \sqrt{\sum_{j=1}^{n_v} v_j^2}$. We also define

$$L_2 := \{w : \|w\|_2 < \infty\},$$
$$L_{2+} := \{w : w \in L_2, \ w(t) \geq 0 \ (\forall t \in [0, \infty))\}.$$

For an operator $H : L_2 \ni w \mapsto z \in L_2$, we define its (standard) L_2 induced norm by

$$\|H\|_2 := \sup_{w \in L_2, \ \|w\|_2 = 1} \|z\|_2. \tag{1}$$

We also define

$$\|H\|_{2+} := \sup_{w \in L_{2+}, \ \|w\|_2 = 1} \|z\|_2. \tag{2}$$

This is a variant of the L_2 induced norm introduced in [5] and referred to as the L_{2+} induced norm in this paper. We can readily see that $\|H\|_{2+} \leq \|H\|_2$.

B. Copositive Programming

A copositive programming problem (COP) is a convex optimization problem in which we minimize a linear objective function over the linear matrix inequality (LMI) constraints on the copositive cone [12]. Even though a COP is a convex optimization problem, it is hard to solve it numerically in general. We summarize the definitions of cones related to the COP and its basics in the appendix section, where the materials there are borrowed in part from [13].

C. Positive Systems

In this paper, we introduce positive filters for the (upper bound) computation of the L_{2+} induced norm of continuous-time LTI systems. The definition of positivity and related results are briefly summarized as follows.

Definition 1: [14] An LTI system is called *internally* positive if its state and output are both nonnegative for any nonnegative input and nonnegative initial state.

Definition 2: [14] An LTI system is called *externally* positive if its output is nonnegative for any nonnegative input under the zero initial state.

Proposition 1: [14] An LTI system with coefficient matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$ and $D \in \mathbb{R}^{l \times m}$ is internally positive if and only if

$$A \in \mathbb{M}^n, \ B \in \mathbb{R}_+^{n \times m}, \ C \in \mathbb{R}_+^{l \times n}, D \in \mathbb{R}_+^{l \times m}.$$

III. L_{2+} Induced Norm Analysis

A. Problem Description

Let us consider the LTI system G given by

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \ x(0) = 0, \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$, and $D \in \mathbb{R}^{n_z \times n_w}$. We assume that the system G is stable, i.e., the matrix A is Hurwitz stable, and the pair (A, B) is controllable. It is well known that the L_2 induced norm $\|G\|_2$ defined by (1) coincides with the H_∞ norm for stable LTI systems and plays an essential role in stability analysis of feedback systems. In this paper, we are interested in computing the L_2 induced norm where the input signal w is restricted to be nonnegative. Namely, we focus on the computation of the L_{2+} induced norm $\|G\|_{2+}$ defined by (2). As noted, it is very clear that $\|G\|_{2+} \leq \|G\|_2$. Here, it is well known that $\|G\|_{2+} = \|G\|_2$ holds if G is externally positive, see, e.g., [8], [9].

B. Motivating Example: L_{2+}-Induced-Norm-Based Small Gain Theorem

Let us assume $n_z = n_w = m$ for G given by (3) and consider the feedback system shown in Fig. 1, where $\Phi : \mathbb{R}^m \mapsto \mathbb{R}_+^m$ is a static nonlinear operator satisfying $\|\Phi\|_2 = 1$. We focus on the stability analysis of this feedback system. Here, note that we have assumed that Φ returns only nonnegative signals. This problem setting typically appears in the stability analysis of recurrent neural networks with activation functions being rectified linear units, see [13], [5].

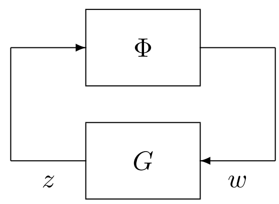


Fig. 1. Nonlinear Feedback System.

Then, from the standard L_2-induced-norm-based small-gain theorem [6], we see that the feedback system shown in Fig. 1 is (well-posed and) globally stable if $\|G\|_2 < 1$. On the other hand, by actively using the nonnegative nature of Φ, it has been shown very recently in [5] that the feedback system shown in Fig. 1 is (well-posed and) globally stable if $\|G\|_{2+} < 1$. As illustrated by this concrete example, the L_{2+}-induced-norm-based small-gain theorem has potential abilities for the stability analysis of feedback systems with nonnegative nonlinearities. This strongly motivates us to establish efficient methods for the (upper bound) computation of the L_{2+} induced norm.

C. Basic Results

The next result forms an important basis of this study.

Proposition 2: [5] For the LTI system G given by (3) and a given $\gamma > 0$, let us consider the following conditions (i) and (ii).

(i) $\|G\|_{2+} \le \gamma$.

(ii) There exist $P \in \mathcal{PSD}^n$ and $Q \in \mathcal{COP}^{n_w}$ such that
$$\begin{bmatrix} PA + A^T P & PB + C^T D \\ * & D^T D - \gamma^2 I_{n_w} + Q \end{bmatrix} \preceq 0. \tag{4}$$

Then we have (i) $\Leftarrow$ (ii).

The copositive matrix variable Q in (4) is referred to as the copositive multiplier in this paper. On the basis of Proposition 2, let us consider the COP:

$$\overline{\gamma} := \inf_{\gamma,P,Q} \text{ subject to } (4),\ P \in \mathcal{PSD}^n,\ Q \in \mathcal{COP}^{n_w}. \tag{5}$$

In relation to this COP, recall that

$$\|G\|_2 = \inf_{\gamma,P} \text{ subject to } (4),\ P \in \mathcal{PSD}^n,\ Q = 0.$$

It follows that $\|G\|_{2+} \le \overline{\gamma} \le \|G\|_2$. Unfortunately, as we have already mentioned, it is hard to solve the COP (5) in general. However, an upper bound of $\overline{\gamma}$ can be computed efficiently by replacing the copositive cone $\mathcal{COP}$ in (5) with the Minkowski sum of the positive semidefinite and nonnegative cones $\mathcal{PSD} + \mathcal{NN}$ as follows:

$$\begin{aligned} &\overline{\overline{\gamma}} := \inf_{\gamma,P,Q} \text{ subject to } (4), \\ &P \in \mathcal{PSD}^n,\ Q \in \mathcal{PSD}^{n_w} + \mathcal{NN}^{n_w}. \end{aligned} \tag{6}$$

Note that this problem is essentially an SDP and hence tractable. We can readily see that $\|G\|_{2+} \le \overline{\gamma} \le \overline{\overline{\gamma}} \le \|G\|_2$ holds.

Up to this point, we have described the basic ideas of the (upper bound) computation of $\|G\|_{2+}$ given in [5]. However, these results are certainly deficient in the following aspects:

(i) In the case where $n_w = 1$, i.e., if the system G has only a single disturbance input, then it is very clear that $\overline{\gamma} = \|G\|_2$. This is because, since $\mathcal{COP}^1 = \mathcal{PSD}^1 = \mathbb{R}_+$, and since the copositive multiplier Q enters in block-diagonal part in (4), we see that the optimal value of Q in COP (5) is zero. Namely, if $n_w = 1$, it is impossible to obtain an upper bound of $\|G\|_{2+}$ which is better than the trivial upper bound $\|G\|_2$ if we directly work on (5).

(ii) In the case where $n_w > 1$, it has been shown by numerical examples in [5] that we can obtain an upper bound $\overline{\overline{\gamma}}$ of $\|G\|_{2+}$ that is strictly better than $\|G\|_2$ (i.e., $\overline{\overline{\gamma}} < \|G\|_2$). However, once we obtain $\overline{\overline{\gamma}}$, we have no way to obtain further better upper bounds than $\overline{\overline{\gamma}}$.

To overcome these difficulties, we provide a new way for the upper bound computation of the L_{2+} induced norm with positive filters in this paper.

Remark 1: In [13], the l_{2+} induced norm for discrete-time operators has been defined in a similar way to (2). In the COP-based treatments of l_{2+} induced norm computation of discrete-time LTI systems, we of course encounter the same difficulties as (i) and (ii) described above. However, in the case of discrete-time systems, we can employ the discrete-time system lifting [15] so that we can artificially increase the number of disturbance inputs. This enables us to employ copositive multipliers of larger size. By means of this lifting-based treatment, we can establish an effective method for the upper bound computation of l_{2+} induced norm, see [13] for details. We finally note that there is no genuine counterpart of the lifting in the continuous-time system setting, and hence we surely need an alternative solution.

IV. Better Upper Bound Computation by Positive Filters

For better upper bound computation of $\|G\|_{2+}$, it is promising to actively use the fact that the input signal w is restricted to be nonnegative. To this end, let us introduce the positive filter given by

$$G_p : \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p w(t),\ x_p(0) = 0, \\ z_p(t) = \begin{bmatrix} I_{n_p} \\ 0_{n_w,n_p} \end{bmatrix} x_p(t) + \begin{bmatrix} 0_{n_p,n_w} \\ I_{n_w} \end{bmatrix} w(t). \end{cases} \tag{7}$$

Here, $A_p \in \mathbb{H}^{n_p} \cap \mathbb{M}^{n_p}$, $B_p \in \mathbb{R}_+^{n_p \times n_w}$. It is clear that the filter G_p is positive from Proposition 1.

We next plug G_p with G and construct the augmented system G_a given by

$$G_a : \begin{cases} \dot{x}_a(t) = A_a x_a(t) + B_a w(t), \\ z(t) = C_a x_a(t) + D_a w(t), \\ z_p(t) = \begin{bmatrix} 0_{n_p,n} & I_{n_p} \\ 0_{n_w,n} & 0_{n_w,n_p} \end{bmatrix} x_a(t) + \begin{bmatrix} 0_{n_p,n_w} \\ I_{n_w} \end{bmatrix} w(t). \end{cases} \tag{8}$$

Here,

$$\begin{aligned} &x_a := \begin{bmatrix} x \\ x_p \end{bmatrix},\ A_a := \begin{bmatrix} A & 0 \\ 0 & A_p \end{bmatrix},\ B_a := \begin{bmatrix} B \\ B_p \end{bmatrix}, \\ &C_a := \begin{bmatrix} C & 0_{n_z,n_p} \end{bmatrix}, D_a := D. \end{aligned} \tag{9}$$

In the above augmented system G_a, it is very important to note that the output z_p is nonnegative for any nonnegative input w. By focusing on this property, we can obtain the first main result of this paper as summarized in the next theorem.

Theorem 1: For the LTI system G given by (3) and a given $\gamma > 0$, let us consider the following conditions (i) and (iii).

(i) $\|G\|_{2+} \le \gamma$.

(iii) There exist $P_a \in \mathbb{S}^{n+n_p}$ and $Q_a \in \mathcal{COP}^{n_p+n_w}$ such that
$$\begin{aligned} &\begin{bmatrix} P_a A_a + A_a^T P_a + C_a^T C_a & P_a B_a + C_a^T D_a \\ B_a^T P_a + D_a^T C_a & D_a^T D_a - \gamma^2 I_{n_w} \end{bmatrix} \\ &+ \begin{bmatrix} 0_{n,n_p+n_w} \\ I_{n_p+n_w} \end{bmatrix} Q_a \begin{bmatrix} 0_{n,n_p+n_w} \\ I_{n_p+n_w} \end{bmatrix}^T \preceq 0. \end{aligned} \tag{10}$$

Then, we have (i) $\Leftarrow$ (iii).

For the proof of this theorem, we need the next lemmas. In the following, we partition $Q_a \in \mathcal{COP}^{n_p+n_w}$ as

$$Q_a = \begin{bmatrix} Q_{a,11} & Q_{a,12} \\ Q_{a,12}^T & Q_{a,22} \end{bmatrix},\ Q_{a,11} \in \mathcal{COP}^{n_p},\ Q_{a,22} \in \mathcal{COP}^{n_w}.$$

Lemma 1: For $A_p \in \mathbb{H}^{n_p} \cap \mathbb{M}^{n_p}$ and $Q_{a,11} \in \mathcal{COP}^{n_p}$, let us consider the unique solution $P_p \in \mathbb{S}^{n_p}$ to the Lyapunov equation

$$P_p A_p + A_p^T P_p + Q_{a,11} = 0. \tag{11}$$

Then, we have $P_p \in \mathcal{COP}^{n_p}$.

Proof of Lemma 1: The proof can readily be done if we note that

$$P_p = \int_0^\infty \exp(A_p^T t) Q_{a,11} \exp(A_p t) dt$$

and $\exp(A_p t) \ge 0$ $(\forall t \ge 0)$ holds since A_p is Metzler. ■

Lemma 2: Suppose $P_a \in \mathbb{S}^{n+n_p}$ satisfies (10) with $Q_a \in \mathcal{COP}^{n_p+n_w}$. Then, we have

$$P_a \in \left\{ P + \begin{bmatrix} 0_{n,n} & 0_{n,n_p} \\ 0_{n_p,n} & P_p \end{bmatrix} : \ P \in \mathcal{PSD}^{n+n_p},\ P_p \in \mathcal{COP}^{n_p} \right\}. \quad (12)$$

Proof of Lemma 2: Suppose $P_a \in \mathbb{S}^{n+n_p}$ satisfies (10) with $Q_a \in \mathcal{COP}^{n_p+n_w}$. Then, it is very clear that there exists $W \in \mathbb{S}_+^{n+n_p}$ such that

$$P_a A_a + A_a^T P_a + C_a^T C_a + \begin{bmatrix} 0_{n,n} & 0_{n,n_p} \\ 0_{n_p,n} & Q_{a,11} \end{bmatrix} + W = 0$$

where $A_a \in \mathbb{H}^{n+n_p}$ from (9). If we regard this equation as the Lyapunov equation with respect to $P_a \in \mathbb{S}^{n+n_p}$, we see from the linearity that $P_a \in \mathbb{S}^{n+n_p}$ can be written as

$$P_a = P + \begin{bmatrix} 0_{n,n} & 0_{n,n_p} \\ 0_{n_p,n} & P_p \end{bmatrix}.$$

Here, $P \in \mathbb{S}_+^{n+n_p}$ is the unique solution to the Lyapunov equation

$$PA_a + A_a^T P + C_a^T C_a + W = 0$$

whereas $P_p \in \mathbb{S}^{n_p}$ is the unique solution to the Lyapunov equation (11). From Lemma 1, we have $P_p \in \mathcal{COP}^{n_p}$ and hence the proof is completed. ■

Now we are ready to prove Theorem 1.

Proof of Theorem 1: For the augmented system G_a, we consider the trajectory of its state x_a for the input $w \in L_{2+}$ with $\|w\|_2 = 1$. From (10), we readily see

$$\begin{bmatrix} x_a(t) \\ w(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_a A_a + A_a^T P_a + C_a^T C_a & P_a B_a + C_a^T D_a \\ B_a^T P_a + D_a^T C_a & D_a^T D_a - \gamma^2 I_{n_w} \end{bmatrix} \begin{bmatrix} x_a(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} x_p(t) \\ w(t) \end{bmatrix}^T Q_a \begin{bmatrix} x_p(t) \\ w(t) \end{bmatrix} \le 0 \ (\forall t \ge 0).$$

From this inequality and (8), we have

$$\frac{d}{dt} x_a(t)^T P_a x_a(t) + z(t)^T z(t) - \gamma^2 w(t)^T w(t) + z_p(t)^T Q_a z_p(t) \le 0 \ (\forall t \ge 0).$$

By integration, we arrive at

$$x_a(T)^T P_a x_a(T) + \int_0^T z(t)^T z(t) - \gamma^2 w(t)^T w(t) dt + \int_0^T z_p(t)^T Q_a z_p(t) dt \le 0 \ (\forall T > 0). \quad (13)$$

Since $Q_a \in \mathcal{COP}^{n_p+n_w}$ and $z_p \in L_{2+}$, we first note that

$$\int_0^T z_p(t)^T Q_a z_p(t) dt \ge 0 \ (\forall T > 0). \quad (14)$$

On the other hand, since x_p is nonnegative in $x_a (= [\ x^T \ x_p^T \ ]^T)$, we see from Lemma 2 that

$$x_a(T)^T P_a x_a(T) \ge 0 \ (\forall T > 0). \quad (15)$$

With these facts in mind, we take the limit $T \to \infty$ in (13) and obtain

$$\int_0^\infty z(t)^T z(t)\, dt - \gamma^2 \int_0^\infty w(t)^T w(t)\, dt \le 0. \quad (16)$$

This clearly shows $\|z\|_2^2 \le \gamma^2 \|w\|_2^2 = \gamma^2$. To summarize, we arrive at the conclusion that

$$\|G\|_{2+} = \sup_{w \in L_{2+}, \|w\|_2 = 1} \|z\|_2 \ \le \ \gamma.$$

This completes the proof. ■

Remark 2: In stark contrast with the standard L_2-induced norm computation case, the Lyapunov certificate $P_a \in \mathbb{S}^{n+n_p}$ that satisfies (10) does not satisfy $P_a \in \mathcal{PSD}^{n+n_p}$ in general. Namely, we have (12).

Remark 3: Even though our proof of Theorem 1 has been done by purely time-domain arguments, it has close relationship with the general IQC theorem [16], [17] basically characterized in frequency domain. Recall that the key conditions in the proof are (14) and (15), and these are related to the motivation of the study in [17]. The key condition (14) is referred to as the hard (finite horizon) IQC in [17], and its weaker condition is called the soft (infinite horizon) IQC. Moreover, even though (15) does hold, the Lyapunov certificate P_a in (10) is not positive semidefinite in general (Lemma 2), and this conforms to the general setting of [17] that deals with indefinite Lyapunov certificates and soft (infinite horizon) IQCs. Therefore, it seems that the current result can be basically subsumed into the general framework of [17]. However, since we have introduced COP multipliers, the relationship of Theorem 1 with [17] is not yet very clear to us. This topic is currently under investigation.

On the basis of Theorem 1, let us consider the following COP and SDP:

$$\overline{\gamma}_a := \inf_{\gamma, P_a, Q_a} \gamma \quad \text{subject to (10)}, \quad P_a \in \mathbb{S}^{n+n_p},\ Q_a \in \mathcal{COP}^{n_p+n_w}, \quad (17)$$

$$\overline{\overline{\gamma}}_a := \inf_{\gamma, P_a, Q_a} \gamma \quad \text{subject to (10)}, \quad P_a \in \mathbb{S}^{n+n_p},\ Q_a \in \mathcal{PSD}^{n_p+n_w} + \mathcal{NN}^{n_p+n_w}. \quad (18)$$

Then, we readily obtain

$$\|G\|_{2+} \le \overline{\gamma}_a \le \overline{\overline{\gamma}}_a. \quad (19)$$

Moreover, we can obtain the next theorem that verifies the effectiveness of the introduction of positive filters in computing better (smaller) upper bounds.

Theorem 2: Let us consider the positive-filter-based upper bounds $\overline{\gamma}_a$ and $\overline{\overline{\gamma}}_a$ of $\|G\|_{2+}$ given respectively by (17) and (18). Then, in relation to the filter-free upper bounds $\overline{\gamma}$ and $\overline{\overline{\gamma}}$ given respectively by (5) and (6), we have

$$\overline{\gamma}_a \le \overline{\gamma} \ (\le \|G\|_2), \quad \overline{\overline{\gamma}}_a \le \overline{\overline{\gamma}} \ (\le \|G\|_2) \quad (20)$$

Proof of Theorem 2: In the following, we prove $\overline{\gamma}_a \le \overline{\gamma}$. The proof for $\overline{\overline{\gamma}}_a \le \overline{\overline{\gamma}}$ follows similarly. To prove $\overline{\gamma}_a \le \overline{\gamma}$, it suffices to show that the condition (10) in (iii) of Theorem 1 holds with $\gamma = \overline{\gamma} + \varepsilon$ for any $\varepsilon > 0$.

We first note from the definition of $\overline{\gamma}$ given by (5) that for any $\varepsilon > 0$ there exist $P \in \mathbb{S}^n$, $Q \in \mathcal{COP}^{n_w}$, and $\varepsilon_1 > 0$

such that

$$\begin{bmatrix} PA+A^TP+C^TC & PB+C^TD & 0 \\ * & D^TD-(\overline{\gamma}+\varepsilon)^2 I_{n_w}+Q & \varepsilon_1 B_p^T P_p \\ * & * & \varepsilon_1(P_pA_p+A_p^TP_p) \end{bmatrix} \preceq 0.$$

Here, $P_p \in \mathbb{S}_{++}^{n_p}$ is the unique solution of the Lyapunov equation

$$P_pA_p + A_p^T P_p + I = 0. \tag{21}$$

By applying a congruence transformation to the preceding inequality, we have

$$\begin{bmatrix} PA+A^TP+C^TC & 0 & PB+C^TD \\ * & \varepsilon_1(P_pA_p+A_p^TP_p) & \varepsilon_1 P_pB_p \\ * & * & D^TD-(\overline{\gamma}+\varepsilon)^2 I_{n_w}+Q \end{bmatrix} \preceq 0.$$

This clearly shows that (10) in (iii) of Theorem 1 holds with $\gamma = \overline{\gamma}+\varepsilon$ and

$$P_a = \begin{bmatrix} P & 0 \\ 0 & \varepsilon_1 P_p \end{bmatrix} \in \mathbb{S}^{n+n_p},\ Q_a = \begin{bmatrix} 0_{n_p,n_p} & 0 \\ 0 & Q \end{bmatrix} \in \mathcal{COP}^{n_p+n_w}.$$

This completes the proof. ■

Theorem 2 shows that for any positive filter G_p the corresponding upper bound $\overline{\gamma}_a$ given by (17) ($\overline{\overline{\gamma}}_a$ given by (18)) is better (no worse) than the filter-free upper bound $\overline{\gamma}$ given by (5) ($\overline{\overline{\gamma}}$ given by (6)). In Section VI, we demonstrate by numerical examples that $\overline{\overline{\gamma}}_a$ can be strictly better than $\overline{\overline{\gamma}}$ as expected.

V. Concrete Construction of Positive Filters

As for the positive filter G_p given by (7), let us consider the specific form given by

$$\begin{aligned} A_p &= A_{p,\alpha,N} := J_{\alpha,N}\otimes I_{n_w} \in \mathbb{R}^{Nn_w\times Nn_w}, \\ B_p &= B_{p,N} := E_N \otimes I_{n_w} \in \mathbb{R}^{Nn_w\times n_w}, \\ J_{\alpha,N} &:= \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \alpha \end{bmatrix} \in \mathbb{R}^{N\times N},\ E_N := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^N \end{aligned} \tag{22}$$

where $\alpha < 0$. In this case, the input-output property of the positive filter G_p in frequency domain is given by

$$Z_p(s) = \begin{bmatrix} \dfrac{1}{(s-\alpha)^N} I_{n_w} \\ \vdots \\ \dfrac{1}{(s-\alpha)} I_{n_w} \\ I_{n_w} \end{bmatrix} W(s).$$

By increasing the degree N of the positive filter G_p given by (7) and (22), we can construct a sequence of COPs in the form of (17) and SDPs in the form of (18). In the following, we denote by $\overline{\gamma}_{a,\alpha,N}$ and $\overline{\overline{\gamma}}_{a,\alpha,N}$ the optimal values of these COPs and SDPs, respectively. In addition, we denote by $A_{a,\alpha,N}$, $B_{a,N}$, $C_{a,N}$, and $D_{a,N}(=D)$ the coefficient matrices of the augmented system G_a given by (8) corresponding to the filter of degree N. Then, regarding the effectiveness of employing higher-degree positive filters in improving upper bounds, we can obtain the next result.

Theorem 3: Let us consider the upper bounds of $\|G\|_{2+}$ given by $\overline{\gamma}_{a,\alpha,N}$ and $\overline{\overline{\gamma}}_{a,\alpha,N}$ that are characterized respectively by (17) and (18) with the positive filter G_p of the form (7) and (22) of degree N. Then, for $N_1 \le N_2$, we have

$$\overline{\gamma}_{a,\alpha,N_2} \le \overline{\gamma}_{a,\alpha,N_1},\quad \overline{\overline{\gamma}}_{a,\alpha,N_2} \le \overline{\overline{\gamma}}_{a,\alpha,N_1}. \tag{23}$$

Proof of Theorem 3: In the following, we prove $\overline{\gamma}_{a,\alpha,N_2} \le \overline{\gamma}_{a,\alpha,N_1}$. The proof for $\overline{\overline{\gamma}}_{a,\alpha,N_2} \le \overline{\overline{\gamma}}_{a,\alpha,N_1}$ follows similarly. To prove $\overline{\gamma}_{a,\alpha,N_2} \le \overline{\gamma}_{a,\alpha,N_1}$, it suffices to show that $\overline{\gamma}_{a,\alpha,N+1} \le \overline{\gamma}_{a,\alpha,N}$ holds for any N. Furthermore, this can be verified by proving that (10) corresponding to the filter of degree $N+1$ holds with $\gamma = \overline{\gamma}_{a,\alpha,N}+\varepsilon$ for any $\varepsilon > 0$.

To this end, we first note from the definition of $\overline{\gamma}_{a,\alpha,N}$ that for any $\varepsilon > 0$ there exist $P_a = P_{a,\alpha,N} \in \mathbb{S}^{n+Nn_w}$ and $Q_a = Q_{a,\alpha,N} \in \mathcal{COP}^{(N+1)n_w}$ such that

$$\begin{aligned} &\begin{bmatrix} P_{a,\alpha,N}A_{a,\alpha,N}+A_{a,\alpha,N}^TP_{a,\alpha,N} & P_{a,\alpha,N}B_{a,N} \\ B_{a,N}^TP_{a,\alpha,N} & -(\overline{\gamma}_{a,\alpha,N}^2+2\overline{\gamma}_{a,\alpha,N}\varepsilon)I_{n_w} \end{bmatrix} \\ &+ \begin{bmatrix} C_{a,N}^T \\ D_{a,N}^T \end{bmatrix} \begin{bmatrix} C_{a,N}^T \\ D_{a,N}^T \end{bmatrix}^T \\ &+ \begin{bmatrix} 0_{n,(N+1)n_w} \\ I_{(N+1)n_w} \end{bmatrix} Q_{a,\alpha,N} \begin{bmatrix} 0_{n,(N+1)n_w} \\ I_{(N+1)n_w} \end{bmatrix}^T \preceq 0. \end{aligned} \tag{24}$$

To proceed, let us define

$$F_N := \begin{bmatrix} I_{n_w} & 0_{n_w,n_w} & \cdots & 0_{n_w,n_w} \end{bmatrix} \in \mathbb{R}^{n_w\times Nn_w}.$$

Then, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\begin{bmatrix} 0_{n,n} & 0 & 0 \\ * & -\varepsilon_2 I_{Nn_w} - \dfrac{\varepsilon_1}{2\alpha}F_N^TF_N & \varepsilon_2 P_pB_p \\ * & * & -\varepsilon^2 I_{n_w} \end{bmatrix} \preceq 0$$

where $P_p \in \mathbb{S}_{++}^{n_p}$ is the unique solution of the Lyapunov equation (21). By summing up the above inequality with (24) and applying the Schur complement argument, we obtain

$$\begin{aligned} &\left[\begin{matrix} P_{a,\alpha,N}^{11}A + A^TP_{a,\alpha,N}^{11} & 0 \\ * & 2\varepsilon_1\alpha I_{n_w} \\ * & * \\ * & * \end{matrix}\right. \\ &\qquad \left.\begin{matrix} P_{a,\alpha,N}^{11}B + P_{a,\alpha,N}^{12}B_p & \\ \varepsilon_1 F_N & 0 \\ \widehat{P}_{a,\alpha,N}^{22}A_{p,\alpha,N} + A_{p,\alpha,N}^T\widehat{P}_{a,\alpha,N}^{22} & P_{a,\alpha,N}^{12T}B + \widehat{P}_{a,\alpha,N}^{22}B_p \\ * & -(\overline{\gamma}_{a,\alpha,N}+\varepsilon)^2 I_{n_w} \end{matrix}\right] \\ &+ \begin{bmatrix} C_{a,N+1}^T \\ D_{a,N+1}^T \end{bmatrix} \begin{bmatrix} C_{a,N+1}^T \\ D_{a,N+1}^T \end{bmatrix}^T \\ &+ \begin{bmatrix} 0_{n,(N+2)n_w} \\ I_{(N+2)n_w} \end{bmatrix} \begin{bmatrix} 0_{n_w,n_w} & 0 \\ 0 & Q_{a,\alpha,N} \end{bmatrix} \begin{bmatrix} 0_{n,(N+2)n_w} \\ I_{(N+2)n_w} \end{bmatrix}^T \preceq 0. \end{aligned} \tag{25}$$

Here,

$$\begin{bmatrix} P_{a,\alpha,N}^{11} & P_{a,\alpha,N}^{12} \\ P_{a,\alpha,N}^{12T} & P_{a,\alpha,N}^{22} \end{bmatrix} := P_{a,\alpha,N},\ P_{a,\alpha,N}^{11} \in \mathbb{S}^n,\ P_{a,\alpha,N}^{22} \in \mathbb{S}^{Nn_w},$$

$$\widehat{P}_{a,\alpha,N}^{22} := P_{a,\alpha,N}^{22} + \varepsilon_2 P_p.$$

Then, (25) shows that (10) corresponding to the filter of degree $N+1$ holds with $\gamma = \overline{\gamma}_{a,\alpha,N} + \varepsilon$ and

$$P_a = P_{a,\alpha,N+1} = \begin{bmatrix} P_{a,\alpha,N}^{11} & 0 & P_{a,\alpha,N}^{12} \\ 0 & \varepsilon_1 I_{n_w} & 0 \\ P_{a,\alpha,N}^{12T} & 0 & \widehat{P}_{a,\alpha,N}^{22} \end{bmatrix} \in \mathbb{S}^{n+(N+1)n_w},$$
$$Q_a = Q_{a,\alpha,N+1} = \begin{bmatrix} 0_{n_w,n_w} & 0 \\ 0 & Q_{a,\alpha,N} \end{bmatrix} \in \mathcal{COP}^{(N+2)n_w}.$$

This completes the proof. ■

Remark 4: From Theorem 3, we see that we can construct a monotonically non-increasing sequence of upper bounds $\{\overline{\overline{\gamma}}_{a,\alpha,N}\}$ of $\|G\|_{2+}$ by increasing the degree N and solving the corresponding SDP (18). Of course this is done at the expense of increased computational burden.

Remark 5: As shown in the appendix section, we can prove that the (primal) SDP (18) and its dual both have interior point solutions (in the case where we employ the positive filter G_p of the form (7) and (22)). Therefore, there is no duality gap between the SDP (18) and its dual, and both have optimal solutions, see [18].

VI. Numerical Examples

In this section, we illustrate the effectiveness of the proposed positive-filter-based method by numerical examples on single- and multi-input systems. We use MOSEK [19] to solve the SDP (18). As for the algorithm implemented in MOSEK, we can expect the execution of reliable computation if the SDP to be solved and its dual both have interior point solutions. Since the SDP (18) and its dual indeed both have interior solutions, we can conclude that we have executed reliable numerical computation in this section.

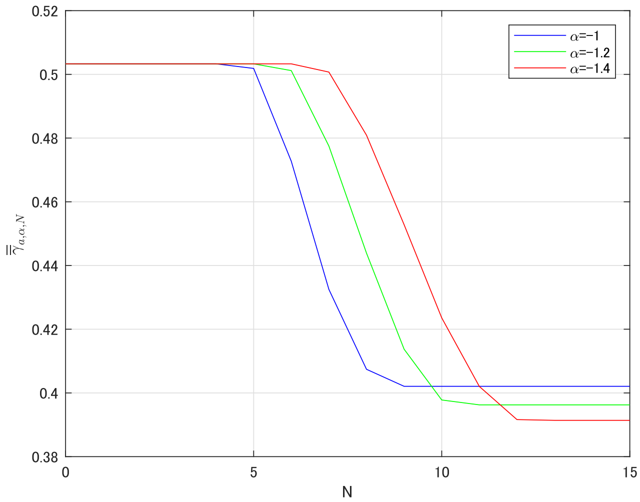
A. Single-Input Case

Let us consider the case where the coefficient matrices of the system (3) are randomly generated and given by

$$A = \begin{bmatrix} -0.09 & 0.28 & 0.46 & -0.48 & -0.05 \\ -0.34 & -0.95 & -0.42 & 0.37 & -0.55 \\ -0.24 & 0.04 & -0.10 & -0.47 & -0.23 \\ 0.30 & 0.29 & 0.02 & -1.59 & 0.57 \\ 0.26 & 0.25 & 0.40 & -0.74 & -0.95 \end{bmatrix}, \ B = \begin{bmatrix} 0.17 \\ 0.40 \\ 0.49 \\ 0.30 \\ -0.69 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.14 & -0.66 & 0.10 & 0.34 & 0.05 \end{bmatrix}, \ D = 0.27.$$

In this case, it turned out that $\|G\|_2 = 0.5033$ $(= \overline{\overline{\gamma}}_{a,\alpha,0})$. Then, for $\alpha \in \{-1, -1.2, -1.4\}$, we constructed positive filters of degree N and then obtained upper bounds $\overline{\overline{\gamma}}_{a,\alpha,N}$ by solving the SDP (18). The results are shown in Fig. 2. We note that $\overline{\overline{\gamma}}_{a,\alpha,N} = \overline{\gamma}_{a,\alpha,N}$ holds up to $N=3$ in this case, since for the SDP (18) we have $Q_a \in \mathcal{PSD}^{N+1} + \mathcal{NN}^{N+1} = \mathcal{COP}^{N+1}$ $(N \le 3)$.

From Fig. 2, the best (least) upper bound turned out to be $\overline{\overline{\gamma}}_{a,\alpha,N} = 0.3914$ with $\alpha = -1.4$ and $N = 15$. For each α, we see that $\overline{\overline{\gamma}}_{a,\alpha,N}$ is monotonically non-increasing with respect to N and this result is surely consistent with Theorem 3.



The Degree of Positive Filters

Fig. 2. The Values of $\overline{\overline{\gamma}}_{a,\alpha,N}$: Upper Bounds of $\|G\|_{2+}$.

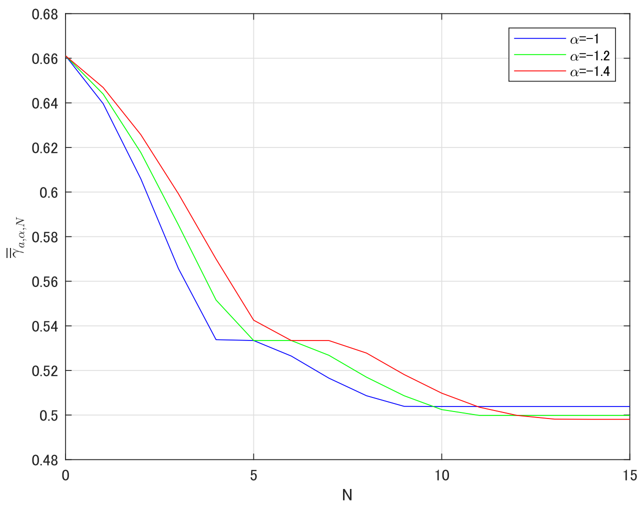
B. Multi-Input Case

We next consider the case where the coefficient matrices of the system (3) are randomly generated and given by

$$A = \begin{bmatrix} -0.11 & -0.15 & 0.18 & 0.15 & -0.10 \\ 0.18 & -0.53 & -0.35 & 0.37 & -0.23 \\ -0.64 & -0.12 & -0.75 & 0.23 & 0.59 \\ 0.34 & -0.03 & 0.13 & -0.47 & -0.67 \\ 0.55 & 0.29 & -0.08 & 0.53 & -0.81 \end{bmatrix}, \ B = \begin{bmatrix} -0.14 & 0.32 \\ -0.76 & -0.42 \\ -0.30 & -0.03 \\ 0.64 & -0.38 \\ -0.12 & 0.17 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.35 & 0.03 & 0.33 & 0.05 & 0.14 \end{bmatrix}, \ D = \begin{bmatrix} 0.43 & 0.23 \end{bmatrix}.$$

In this case, it turned out that $\|G\|_2 = 0.6995$. On the other hand, by solving the SDP (6) shown in [5], we obtained $\overline{\overline{\gamma}} = 0.6611$ $(= \overline{\overline{\gamma}}_{a,\alpha,0})$. We note that $\overline{\overline{\gamma}} = \overline{\gamma}$ holds in this case since for the SDP (6) we have $Q \in \mathcal{PSD}^2 + \mathcal{NN}^2 = \mathcal{COP}^2$.

Then, for $\alpha \in \{-1, -1.2, -1.4\}$, we constructed positive filters of degree N and then obtained upper bounds $\overline{\overline{\gamma}}_{a,\alpha,N}$ by solving the SDP (18). The results are shown in Fig. 3. The best (least) upper bound turned out to be $\overline{\overline{\gamma}}_{a,\alpha,N} = 0.4981$ with $\alpha = -1.4$ and $N = 15$. For each α, we see that $\overline{\overline{\gamma}}_{a,\alpha,N}$ is monotonically non-increasing with respect to N and this result is again consistent with Theorem 3.



The Degree of Positive Filters

Fig. 3. The Values of $\overline{\overline{\gamma}}_{a,\alpha,N}$: Upper Bounds of $\|G\|_{2+}$.

VII. Conclusion and Future Works

In this paper, we considered the (upper bond) computation of the L_{2+} induced norm for continuous-time LTI systems. To obtain better (smaller) upper bounds, we introduced positive filters and reduced the upper bound computation problem into a COP. Then, by applying inner approximation to the COP cone, we derived a numerically tractable SDP for the upper bound computation. By numerical examples, we showed the effectiveness of the proposed positive-filter-based method to obtain better (smaller) upper bounds.

In this paper we just focused on the upper bound computation of the L_{2+} induced norm. It is nonetheless true that we cannot say anything on the conservatism of the obtained upper bounds if we merely rely on upper bound computation. Therefore it is desirable to compute lower bounds of L_{2+} induced norm efficiently. Our upcoming results on such lower bound computation will be reported elsewhere in a near future.

Appendix

Basics about Copositive Programming

We first review the definitions and the properties of convex cones related to the COP.

Definition 3: [20] The definitions of proper cones $\mathcal{PSD}^n$, $\mathcal{COP}^n$, $\mathcal{CP}^n$, $\mathcal{NN}^n$, and $\mathcal{DNN}^n$ in $\mathbb{S}^n$ are as follows.

1) $\mathcal{PSD}^n := \{P \in \mathbb{S}^n : \ \forall x \in \mathbb{R}^n,\ x^T P x \geq 0\} = \{P \in \mathbb{S}^n : \ \exists B \text{ s.t. } P = BB^T\}$ is called *the positive semidefinite cone*.
2) $\mathcal{COP}^n := \{P \in \mathbb{S}^n : \ \forall x \in \mathbb{R}_+^n,\ x^T P x \geq 0\}$ is called *the copositive cone*.
3) $\mathcal{CP}^n := \{P \in \mathbb{S}^n : \ \exists B \geq 0 \text{ s.t. } P = BB^T\}$ is called *the completely positive cone*.
4) $\mathcal{NN}^n := \{P \in \mathbb{S}^n : \ P \geq 0\}$ is called *the nonnegative cone*.
5) $\mathcal{PSD}^n + \mathcal{NN}^n := \{P + Q : \ P \in \mathcal{PSD}^n,\ Q \in \mathcal{NN}^n\}$. This is the Minkowski sum of the positive semidefinite cone and the nonnegative cone.
6) $\mathcal{DNN}^n := \mathcal{PSD}^n \cap \mathcal{NN}^n$ is called *the doubly nonnegative cone*.

From Definition 3, we clearly see that the following inclusion relationships hold:

$$\mathcal{CP}^n \subset \mathcal{DNN}^n \subset \mathcal{PSD}^n \subset \mathcal{PSD}^n + \mathcal{NN}^n \subset \mathcal{COP}^n, \quad (26)$$

$$\mathcal{CP}^n \subset \mathcal{DNN}^n \subset \mathcal{NN}^n \subset \mathcal{PSD}^n + \mathcal{NN}^n \subset \mathcal{COP}^n. \quad (27)$$

In particular, when $n \leq 4$, it is known that $\mathcal{COP}^n = \mathcal{PSD}^n + \mathcal{NN}^n$ and $\mathcal{CP}^n = \mathcal{DNN}^n$ hold [20]. On the other hand, as for the duality of these cones, $\mathcal{COP}^n$ and $\mathcal{CP}^n$ are dual to each other, $\mathcal{PSD}^n + \mathcal{NN}^n$ and $\mathcal{DNN}^n$ are dual to each other, and $\mathcal{PSD}^n$ and $\mathcal{NN}^n$ are self-dual. It is also well known that the interiors of the cones $\mathcal{PSD}^n$ and $\mathcal{NN}^n$can be characterized by

$$\begin{aligned}(\mathcal{PSD}^n)^\circ &= \{P \in \mathbb{S}^n : \ \forall x \in \mathbb{R}^n \backslash \{0\},\ x^T P x > 0\},\\ (\mathcal{NN}^n)^\circ &= \{P \in \mathbb{S}^n : \ P > 0\}.\end{aligned}$$

As noted, the COP is a convex optimization problem on the copositive cone, and its dual is a convex optimization problem on the completely positive cone. As mentioned in [12], the problem to determine whether a given symmetric matrix is copositive or not is a co-NP complete problem, and the problem to determine whether a given symmetric matrix is completely positive or not is an NP-hard problem. Therefore, it is hard to solve COP numerically in general. However, since the problem to determine whether a given matrix is in $\mathcal{PSD}+\mathcal{NN}$ or in $\mathcal{DNN}$ can readily be reduced to SDPs, we can numerically solve the convex optimization problems on the cones $\mathcal{PSD}+\mathcal{NN}$ and $\mathcal{DNN}$ easily. Moreover, when $n \leq 4$, it is known that $\mathcal{COP}^n = \mathcal{PSD}^n + \mathcal{NN}^n$ and $\mathcal{CP}^n = \mathcal{DNN}^n$ as stated above, and hence those COPs with $n \leq 4$ can be reduced to SDPs.

On the Existence of Interior Point Solutions for the SDP (18) and Its Dual

We first note that the dual of the SDP (18) is given by

$$\sup_Z \ \text{trace}\left(\begin{bmatrix} C_{a,N}^T \\ D_{a,N}^T \end{bmatrix}^T Z \begin{bmatrix} C_{a,N}^T \\ D_{a,N}^T \end{bmatrix}\right) \quad \text{subject to}$$

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ * & Z_{22} & Z_{23} \\ * & * & Z_{33} \end{bmatrix} \in \mathcal{PSD}^{n+n_p+n_w}, \quad (28a)$$

$$\text{trace}(Z_{33}) = 1, \quad (28b)$$

$$A_a Z_a + B_a Z_b^T + (A_a Z_a + B_a Z_b^T)^T = 0, \quad (28c)$$

$$Z_c \geq 0, \quad (28d)$$

where

$$\begin{aligned} Z_a &:= \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} \in \mathcal{PSD}^{n+n_p}, \\ Z_b &:= \begin{bmatrix} Z_{13} \\ Z_{23} \end{bmatrix} \in \mathbb{R}^{(n+n_p)\times n_w}, \\ Z_c &:= \begin{bmatrix} Z_{22} & Z_{23} \\ * & Z_{33} \end{bmatrix} \in \mathcal{PSD}^{n_p+n_w}. \end{aligned}$$

On the existence of the interior point solutions for the (primal) SDP (18) and its dual (28), we can establish the following two theorems.

Theorem 4: The SDP (18) has an interior point solution.

Proof of Theorem 4: Let us denote by $P_0 \in \mathbb{S}_{++}^{n+n_p}$ the unique solution of the Lyapunov equation

$$P_0 A_a + A_a^T P_0 + C_a^T C_a + 2 I_{n+n_p} = 0.$$

In addition, with sufficiently small $\varepsilon > 0$ we let

$$Q_0 = I_{n_p+n_w} + \varepsilon \mathbf{1}_{n_p+n_w} \mathbf{1}_{n_p+n_w}^T.$$

Then, it is clear that

$$P_0 \in \mathbb{S}^{n+n_p},\ Q_0 \in (\mathcal{PSD}^{n_p+n_w})^\circ + (\mathcal{NN}^{n_p+n_w})^\circ.$$

Furthermore, if we let $\gamma > 0$ sufficiently large, we can confirm that (10) with $\preceq$ being replaced by $\prec$ holds with $P_a = P_0$ and $Q = Q_0$. This clearly shows that the SDP (18) has an interior point solution. ∎

Theorem 5: The dual SDP (28) has an interior point solution.

We need the next lemma for the proof of Theorem 5.

Lemma 3: For $A_p \in \mathbb{R}^{Nn_w \times Nn_w}$ and $B_p \in \mathbb{R}^{Nn_w \times n_w}$ given by (22), the unique solution $\widehat{Z}_p$ to the following Lyapunov equation satisfies $\widehat{Z}_p > 0$.

$$A_p\widehat{Z}_p + \widehat{Z}_p A_p^T + B_p \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T B_p^T = 0. \tag{29}$$

Proof of Lemma 3: From the well-known analytic expression for the solution of Lyapunov equation, we have

$$\begin{aligned}\widehat{Z}_p &= \int_0^\infty \exp(A_p t) B_p \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T B_p^T \exp(A_p^T t)dt \\ &= \int_0^\infty \begin{bmatrix} \frac{t^{N-1}}{(N-1)!}\exp(\alpha t) I_{n_w} \\ \vdots \\ \exp(\alpha t) I_{n_w}\end{bmatrix} \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T \begin{bmatrix} \frac{t^{N-1}}{(N-1)!}\exp(\alpha t) I_{n_w} \\ \vdots \\ \exp(\alpha t) I_{n_w}\end{bmatrix}^T dt.\end{aligned}$$

This clearly shows that $\widehat{Z}_p > 0$. ■

We are now ready to prove Theorem 5.

Proof of Theorem 5: We first denote by $Z_{a,0} \in \mathbb{S}_+^{n+n_p}$ the unique solution to the Lyapunov equation given below that is obtained by substituting $Z_b = B_a$ in (28c):

$$A_a Z_{a,0} + Z_{a,0} A_a^T + 2B_a B_a^T = 0.$$

Then, since the pair (A_a, B_a) is controllable, we see $Z_{a,0} \succ 0$. With this fact in mind, we next substitute

$$Z_b = B_a + \begin{bmatrix} 0_{n,n_w} \\ \varepsilon \mathbf{1}_{n_p}\mathbf{1}_{n_w}^T \end{bmatrix} =: Z_{b,\varepsilon} \left(=: \begin{bmatrix} \widehat{Z}_{13} \\ \widehat{Z}_{23}\end{bmatrix}\right) \tag{30}$$

in (28c) with sufficiently small $\varepsilon > 0$ and consider the resulting Lyapunov equation:

$$A_a Z_{a,\varepsilon} + Z_{a,\varepsilon} A_a^T + B_a Z_{b,\varepsilon}^T + Z_{b,\varepsilon} B_a^T = 0. \tag{31}$$

Then, for the unique solution $Z_{a,\varepsilon} \in \mathbb{S}^{n+n_p}$ to this Lyapunov equation, we see that $Z_{a,\varepsilon} \succ 0$ holds for sufficiently small $\varepsilon > 0$ since $Z_{a,0} \succ 0$. If we partition $Z_{a,\varepsilon} \succ 0$ as

$$Z_{a,\varepsilon} =: \begin{bmatrix} \widehat{Z}_{11} & \widehat{Z}_{12} \\ * & \widehat{Z}_{22}\end{bmatrix} \succ 0,\ \widehat{Z}_{22} \in \mathcal{PSD}^{n_p}, \tag{32}$$

we see that the $(2,2)$-block of (31) can be written as

$$\begin{aligned}&A_p \widehat{Z}_{22} + \widehat{Z}_{22} A_p \\ &+B_p(B_p + \varepsilon \mathbf{1}_{n_p}\mathbf{1}_{n_w}^T)^T + (B_p + \varepsilon \mathbf{1}_{n_p}\mathbf{1}_{n_w}^T)B_p^T = 0.\end{aligned}$$

If we compare the above equation with (29) in Lemma 3, we see

$$\begin{aligned}&B_p(B_p + \varepsilon \mathbf{1}_{n_p}\mathbf{1}_{n_w}^T)^T + (B_p + \varepsilon \mathbf{1}_{n_p}\mathbf{1}_{n_w}^T)B_p^T \\ &\geq \varepsilon(B_p \mathbf{1}_{n_w}\mathbf{1}_{n_p}^T + \mathbf{1}_{n_p}\mathbf{1}_{n_w}^T B_p^T) \\ &\geq \varepsilon(B_p \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T B_p^T + B_p \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T B_p^T) \\ &= 2\varepsilon B_p \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T B_p^T.\end{aligned}$$

It follows that

$$\widehat{Z}_{22} \geq 2\varepsilon \widehat{Z}_p > 0. \tag{33}$$

Finally, if we let

$$Z_{\nu,33} := \nu I_{n_w} + \mathbf{1}_{n_w}\mathbf{1}_{n_w}^T (= \widehat{Z}_{33}) \tag{34}$$

with sufficiently large $\nu > 0$, we see from (32) that

$$\widehat{Z} = \begin{bmatrix} \widehat{Z}_{11} & \widehat{Z}_{12} & \widehat{Z}_{13} \\ * & \widehat{Z}_{22} & \widehat{Z}_{23} \\ * & * & \widehat{Z}_{33}\end{bmatrix} \succ 0.$$

In addition, if we define

$$Z := \frac{1}{\mathrm{trace}(\widehat{Z}_{33})}\widehat{Z},$$

we can conclude that this Z satisfies $Z \succ 0$, (28b), (28c), and $Z_c > 0$. To summarize, the dual SDP (28) has an interior point solution. ■

References

[1] C. W. Anderson, P. M. Young, M. R. Buehner, J. N. Knight, K. A. Bush, and D. C. Hittle. Robust reinforcement learning control using integral quadratic constraints for recurrent neural networks. *IEEE Transactions on Neural Networks*, Vol. 18, No. 4, pp. 993–1002, 2007.

[2] H. Yin, P. Seiler, and M. Arcak. Stability analysis using quadratic constraints for systems with neural network controllers. *IEEE Transactions on Automatic Control*, to appear.

[3] M. Revay, R. Wang, and I. R. Manchester. A convex parameterization of robust recurrent neural networks. *IEEE Control Systems Letters*, Vol. 5, No. 4, pp. 1363–1368, 2021.

[4] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, Vol. 42, No. 6, pp. 819–830, 1997.

[5] H. Motooka and Y. Ebihara. L_2 induced norm analysis for nonnegative input signals and its application to stability analysis of recurrent neural networks (in Japanese). *Transactions of the ISCIE, to appear*, 2022.

[6] H. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

[7] C. Briat. Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization. *International Journal of Robust and Nonlinear Control*, Vol. 23, No. 17, pp. 1932–1954, 2013.

[8] T. Tanaka and C. Langbort. The bounded real lemma for internally positive systems and H_∞ structured static state feedback. *IEEE Transactions on Automatic Control*, Vol. 56, No. 9, pp. 2218–2223, 2011.

[9] A. Rantzer. On the Kalman-Yakubovich-Popov lemma for positive systems. *IEEE Transactions on Automatic Control*, Vol. 61, No. 5, pp. 1346–1349, 2016.

[10] Y. Ebihara, D. Peaucelle, and D. Arzelier. Analysis and synthesis of interconnected positive systems. *IEEE Transactions on Automatic Control*, Vol. 62, No. 2, pp. 652–667, 2017.

[11] T. Kato, Y. Ebihara, and T. Hagiwara. Analysis of positive systems using copositive programming. *IEEE Control Systems Letters*, Vol. 4, No. 2, pp. 444–449, 2020.

[12] M. Dür. Copositive programming - a survey. In M. Diehl, F. Glineur, E. Jarlebring, and W. Michiels, editors, *Recent Advances in Optimization and Its Applications in Engineering*, pp. 3–20. Springer, 2010.

[13] Y. Ebihara, H. Waki, V. Magron, N. H. A. Mai, D. Peaucelle, and S. Tarbouriech. l_2 induced norm analysis of discrete-time LTI systems for nonnegative input signals and its application to stability analysis of recurrent neural networks. *European Journal of Control*, to appear 2021.

[14] L. Farina and S. Rinaldi. *Positive Linear Systems: Theory and Applications*. John Wiley and Sons, Inc., 2000.

[15] S. Bittanti and P. Colaneri. Analysis of discrete-time linear periodic systems. In Cornelius T. Leondes, editor, *Control and Dynamic Systems*, Vol. 78, pp. 313–339. Academic Press, New York, 1996.

[16] C. W. Scherer and I. E. Köse. Robustness with dynamic IQCs: An exact state-space characterization of nominal stability with applications to robust estimation. *Automatica*, Vol. 44, No. 7, pp. 1666–1675, 2008.

[17] C. W. Scherer and J. Veenman. Stability analysis by dynamic dissipation inequalities: On merging frequency-domain techniques with time-domain conditions. *Systems and Control Letters*, Vol. 121, pp. 7–15, 2018.

[18] E. Klerk. *Aspects of Semidefinite Programming*. Kluwer Academic Publishers, 2002.

[19] https://www.mosek.com/.

[20] A. Berman and N. Shaked-Monderer. *Completely positive matrices*. World Scientific Publishing, Singapore, 2003.